Chapter 6

Teaching Practices of Instructors in Abstract Algebra[1]

Fatma Sümeyye Uçak[2]

Tuğba Horzum[3]

Abstract

Teaching abstract algebra presents considerable challenges owing to its theoretical nature, necessitating a balance between conceptual understanding and effective teaching strategies. Students frequently encounter difficulties with abstraction, which is essential in mathematics education. Consequently, instructors are required to implement targeted teaching strategies to improve understanding. This study examines the teaching practices of instructors in teaching abstract algebra, emphasizing their approaches to addressing student challenges, organizing content, and employing assessment strategies to enhance learning outcomes. This study investigates the teaching practices utilized by university instructors in Türkiye to facilitate abstract algebra learning. It focuses on the ways in which instructors modify their teaching approaches to meet students' needs, organize course content, and incorporate assessment methods to improve conceptual understanding, as well as their strategies for utilizing and developing abstract algebra curricula. A qualitative case study methodology was utilized, incorporating semi-structured interviews with four university instructors. Thematic content analysis was employed to classify data according to essential components of pedagogical content knowledge, such as student understanding, content knowledge, teaching methods, assessment strategies, and curriculum knowledge. The results demonstrate that instructors primarily employ lecture-based methods,

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2 Mathematics Teacher, Ministry of National Education, Konya, Türkiye, fatmasmyye@gmail.com, https://orcid.org/0000-0003-0814-567X

3 Corresponding Author, Prof. Dr. Necmettin Erbakan University, Konya, Türkiye, thorzum@gmail.com, https://orcid.org/0000-0003-0630-4518

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augmented by question-and-answer techniques and organized examples. Emphasis is placed on conceptual connections and assessments of prior knowledge to address student misconceptions. Instructors identify curriculum limitations, such as inadequate course hours, which restrict comprehensive engagement with abstract concepts. Assessment strategies emphasize the identification of misconceptions via targeted questioning and open-ended problem-solving tasks. This study enhances pedagogical discourse on abstract algebra by examining how instructors utilize pedagogical content knowledge to tackle student challenges. This underscores the necessity for alternative pedagogical approaches, including interactive learning and the integration of technology, to enhance comprehension. The study offers insights into improving abstract algebra instruction, recommending curriculum modifications, varied teaching strategies, and assessment techniques that foster deeper learning. The results can guide faculty development initiatives focused on enhancing abstract algebra teaching methods.

1. Introduction

Numerous individuals choose mathematics or mathematics education programs because of their enthusiasm and aptitude for the subject; abstract algebra, as a core topic in these programs, provides a foundational basis for subsequent mathematics courses. Nonetheless, the rote-based teaching of abstract algebra and the restricted number of students who attain profound comprehension are troubling (Cnop & Grandsard, 1998). This situation has prompted research addressing questions: "What is the role of abstract algebra in teacher education?", "What are the most effective teaching methods for abstract algebra?", and "What challenges hinder the effective teaching of abstract algebra?" (Agustyaningrum et al., 2021; Álvarez et al., 2022; Rupnow et al., 2021; Simpson & Stehlíková, 2006; Veith et al., 2022a).

Research in abstract algebra has predominantly focused on concepts including groups, rings, fields, permutations, isometries, Cayley tables, polynomial roots, and solving equations in $\mathbb{Z}_n$, accompanied by relevant examples and proofs (Álvarez et al., 2022; Çetin & Dikici, 2021; Fukawa-Connelly, 2014; Veith et al., 2022a; Wheeler & Champion, 2013). Findell (2001) asserts that abstract algebra consolidates various mathematical systems inside a common axiomatic structure, whilst Agustyaningrum et al. (2021) underscore that this characteristic improves students' capacity for mathematical abstraction. The axiomatic nature of abstract algebra is recognized as challenging in the teaching and learning process (Agustyaningrum et al., 2021; Leron & Dubinsky, 1995; Nardi, 2000).

These issues cause certain students in mathematics-related programs to disengage from mathematics (Clark et al., 1997; Subedi, 2020).

The difficulties in teaching abstract algebra arise from its intrinsic complexity, the necessity for a robust conceptual basis, and the instructional methods utilized, with the instructor's influence being a considerable contributor to these challenges (Agustyaningrum et al., 2021; Gnawali, 2024; Johnson et al., 2018; Subedi, 2020; Veith et al., 2022a). Challenges encountered by instructors concerning in-class and out-of-class activities, instructional methodologies, and evaluation techniques are thoroughly reported (Barbut, 1987). This study seeks to elucidate instructors' pedagogical practices for understanding their students, delivering content, employing pedagogical techniques, and implementing the curriculum.

2. Theoretical Perspective

2.1. Pedagogical Content Knowledge

Substantial innovations have been implemented in teacher education in Türkiye during the last 30 years to cultivate qualified teachers. Enhancing the quality of the teaching profession is achievable by identifying general and subject-specific competencies and fostering their growth through pre-service and in-service training programs (Ministry of National Education [MoNE], 2017). However, conflicts between subject matter knowledge and pedagogical knowledge endure, requiring a balance between in-depth teaching in subject matter and the significance of pedagogical knowledge. Thus, the necessity of pedagogical content knowledge (PCK), which combines content knowledge with pedagogical knowledge, has been highlighted (Borko et al., 1992; Ma, 2010).

Shulman (1986) was one of the initial researchers to investigate teacher behaviors, the essential knowledge teachers must have, and the role this knowledge plays in the teaching process. They assert that pedagogical content knowledge includes teaching methods tailored to specific disciplines, including mathematics, science, and language, as opposed to broad educational principles. This knowledge encompasses presentations, illustrations, examples, analogies, models, and strategies that facilitate comprehension of the subject matter (Shulman, 1987). They asserted that this knowledge must be robust for effective teaching. Researchers concur that PCK comprises components including knowledge of students' understanding, content, teaching methods and techniques, assessment and evaluation, and curriculum (Chan, 2022; Park & Oliver, 2008).

Implementing strategies to understand students is essential for instructors (Park & Oliver, 2008). This involves being aware of students' prior knowledge, misconceptions, and learning difficulties (Ball et al., 2008; Park & Oliver, 2008; Shulman, 1986, 1987). Learning involves the synthesis of new knowledge with pre-existing knowledge (Manandhar & Sharma, 2021; Soto et al., 2024). Therefore, assessing prior knowledge is critical for determining the necessity of an alternative knowledge structure (Simonsmeier et al., 2022). Identifying the gap between existing and target knowledge structures allows for the appropriate planning of instruction (An et al., 2004). Moreover, anticipating potential difficulties or errors that students may encounter and implementing proactive measures enhances the learning process (Fennema & Franke, 1992; Shulman, 1986, 1987). Moreover, anticipating potential difficulties or errors that students may encounter and implementing proactive measures improves the learning process (Fennema & Franke, 1992; Shulman, 1986, 1987). In courses like abstract algebra, where concepts are interconnected, misconceptions are unavoidable. Instructors must have the requisite knowledge and skills to mitigate these misconceptions (Shulman, 1986). In this context, instructors' knowledge of content presentation is crucial.

Shulman's studies (1986, 1987) highlight the necessity for teachers to utilize subject-specific representations, models, and effective examples in their instructional practices. The complexity of abstract algebra necessitates that instructors choose methods that promote meaningful learning experiences, catering to the diverse needs of students (Ball et al., 2008; Johnson et al., 2019; Rensaa et al., 2021; Stalder, 2023). Rensaa et al. (2021) suggest that mathematics students should concentrate on abstract structures and proofs, whereas engineering students should emphasize concrete applications. Additionally, Stalder (2023) underscores the importance of paradigmatic examples in fostering abstraction. Johnson et al. (2019) attribute the adaptability of abstract algebra in extracurricular pedagogies to curricular innovations and a lack of constraints, while Simpson and Stehlíková (2006) highlight the benefit of redefining representations in enhancing structural understanding. Gnawali (2024) highlights the efficacy of the axiomatic approach in elucidating the connections between abstract structures and their properties, whereas Fukawa-Connelly (2014) claims that instructors should move beyond just illustrating the proof process. Barbut (1987) address the influence of group work utilizing worksheets, while Cnop and Grandsard (1998) underscore the advantages of short tasks for small groups and the incorporation of home materials in facilitating abstract algebra learning. In conclusion, effective teaching of abstract algebra necessitates addressing students' needs through varied strategies and employing methods that link abstract and concrete concepts.

Teaching methods and techniques are vital for efficiently delivering content and meeting the varied needs of students (Capaldi, 2014; Johnson et al., 2018). This knowledge includes selecting instructional approaches, adapting to learning styles, fostering engagement, and accounting for individual differences (Soto et al., 2024). Inquiry-based learning strategies are recognized for their efficacy in enhancing student engagement and understanding in demanding subjects like abstract algebra (Haider & Andrews-Larson, 2022). Research demonstrates the efficacy of multimodal learning strategies and underscores the necessity of designing instructional processes that cater to individual differences to enhance learning (Capaldi, 2014; Durkin et al., 2021). Differentiated strategies can enhance engagement by emphasizing individual strengths (Li, 2023). Technology integration facilitates the adaptation of teaching methods and improves motivation and learning skills (Fortes, 2016; Mrope, 2024; Okur et al., 2011). Developing flexible materials that can adapt to environmental factors is important (Sari & Dimas, 2022). In conclusion, knowledge of teaching methods supports anticipating learning barriers, developing strategies, and helping students achieve their goals.

The assessment and evaluation of student learning in abstract algebra are essential due to the inherent challenges and misconceptions associated with the subject (Veith et al., 2022a). The multiple-choice and written exams prevalent in the Turkish education system urge rote learning and inadequately address misconceptions in the theoretical basis of abstract algebra (Alam & Mohanty, 2024; Subedi, 2020; Veith et al., 2022c). While students may understand the definitions of algebraic structures, they frequently encounter difficulties in applying these concepts to problem-solving, underscoring the necessity for methods that evaluate higher-order thinking skills (Subedi, 2020). Targeted support and effective feedback in critical areas enhance both student performance and teaching efficacy (An et al., 2004; Tanışlı, 2013). Stalder (2023) asserts that suitable feedback enhances comprehension of abstract concepts, whereas Grassl and Mingus (2007) underscore the utility of constructive feedback in areas like groups, rings, and fields within abstract algebra. Veith et al. (2022a) showed that students' expression of abstract algebra concepts in their own terminology reflects their cognitive processes. Formative assessments are essential for identifying misconceptions and modifying instructional strategies (Johnson et al., 2019). Gnawali (2024) emphasizes that an axiomatic approach in abstract algebra fosters a profound conceptual understanding and promotes the ongoing implementation of formative assessments and feedback to sustain student learning.

Abstract algebra instructors need to have a comprehensive understanding of the curriculum's content and structure. Curriculum knowledge constitutes a critical aspect of teacher expertise and enables the development of meaningful learning experiences (Ball et al., 2008; Findell, 2001; Shulman, 1986). This expertise aids students in achieving a deeper comprehension of mathematical concepts and enhances their confidence in learning.

3. Related Literature

3.1. Teaching and Learning Abstract Algebra

Abstract algebra is a mathematical discipline focused on algebraic structures, including groups, rings, and fields, necessitating logical reasoning and abstract thinking because of its abstract nature (Amelia & Effendi, 2020; Wasserman, 2016). The absence of concrete representations for abstract structures poses obstacles for students in comprehending and applying concepts, resulting in both procedural and conceptual difficulties when moving from algebraic operations to broader concepts (Gnawali, 2024; Subedi, 2020). Research indicates that concepts in abstract algebra are fundamental to the principles of mathematics; however, oversimplification may adversely affect students' comprehension (Findell, 2001; Schubert et al., 2013). These challenges highlight the importance of students understanding the relationships among algebraic structures; however, this understanding can be difficult to achieve without sufficient instructional support (Veith et al., 2022a). Leron and Dubinsky (1995) contend that, regardless of instructional quality, student success is contingent upon their preparedness and learning efforts. Pedagogical approaches in abstract algebra must seek to connect prior knowledge with the abstract concepts to be acquired (Capaldi, 2014; Johnson et al., 2018; Manandhar & Sharma, 2021). Research highlights the significance of effective teaching strategies, supportive examples, and technology integration (Manandhar & Sharma, 2021; Mrope, 2024; Okur et al., 2011; Stalder, 2023). Instructors can enhance understanding and engagement in abstract algebra through the use of varied strategies, examples, and innovative methods (Booth et al., 2013; Booth et al., 2015; Capaldi, 2014; Durkin et al., 2021; Fukawa-Connelly et al., 2016). These approaches are crucial for facilitating the learning process and addressing the limitations of conventional methods (Litke, 2020; Veith et al., 2022a). Nevertheless, challenges in teaching abstract algebra are widely acknowledged.

The challenges faced in teaching abstract algebra have motivated instructors to devise innovative solutions. These solutions encompass constructivist approaches (Larsen et al., 2013; Okur et al., 2011), the application of visual representations and concrete examples (Manandhar & Sharma, 2021; Soto et al., 2024), in addition to inquiry-based and collaborative strategies (Khasawneh et al., 2023). Targeted support to address misconceptions has been demonstrated to improve comprehension and performance in abstract algebra (Ndemo & Ndemo, 2018). Technology, especially computer algebra systems, dynamic illustrations, and interactive experiences, enhances learning by making abstract concepts accessible (Velychko et al., 2019).

Studies examining the pedagogical practices of instructors in abstract algebra emphasize the significance of content knowledge and instructional strategies for effective teaching (Fukawa-Connelly, 2012, 2014; Mora et al., 2021). Instructors possessing an in-depth knowledge of abstract algebra demonstrate greater success in resolving students' challenges and misconceptions (Litke et al., 2020; Subedi, 2020). Further studies are necessary to elucidate the specific instructional practices in abstract algebra courses, as this understanding is crucial for enhancing teaching quality and developing professional development programs (Veith et al., 2022a).

4. Methodology

4.1. Research Design

This qualitative case study examines instructors' experiences related to their teaching practices in abstract algebra. Merriam and Tisdell (2016) characterizes qualitative case studies as a method for the in-depth examination and analysis of a particular group or phenomenon. This study concentrates on instructors experienced in the methodologies utilized for teaching abstract algebra.

4.2. Participants

This research involved four teachers teaching Abstract Algebra at universities in Türkiye, with participant characteristics provided in Figure 1. Participants were chosen voluntarily, and pseudonyms—Instructor1, Instructor2, Instructor3, and Instructor4— were employed to maintain their privacy.

	Education level	Experience	Lessons taught at
Instructor1 Male	Bachelor's degree: Mathematics teaching program Master's degree: Algebra and number theory Doctoral degree: Algebra and number theory	24 years' experience Professor Since 2012, he has been teaching abstract algebra. He wrote articles on algebra and number theory, and mathematics education.	Undergraduate level: Linear algebra, Number systems and algebraic structures, Applications of linear algebra, Abstract algebra Postgraduate level: Maple package program supported teaching linear algebra Teaching algebraic concepts, Applications of linear algebra Matrix theory and maple package program
Instructor2 Male	Bachelor's degree: Mathematics program Master's degree: Algebra and number theory Doctoral degree: Algebra and number theory	11 years' experience Assistant professor Since 2015, he has been teaching abstract algebra. He wrote articles on algebra and number theory.	Undergraduate level: Linear algebra, Elementary number theory, Analytical geometry, Differential equations, Abstract algebra Postgraduate level: *Pell and Pell-Lucas number sequence*
Instructor3 Female	Bachelor's degree: Mathematics teaching program Master's degree: Algebra and number theory Doctoral degree: Algebra and number theory	17 years' experience Associate Professor Since 2015, she has been teaching abstract algebra. She wrote articles on algebra and number theory, and mathematics education.	In addition to various theoretical courses including abstract algebra at undergraduate and postgraduate level, she has conducted courses such as special teaching methods, instructional technologies and material design in mathematics education.
Instructor4 Male	Bachelor's degree: Mathematics teaching program Master's degree: Algebra and number theory Doctoral degree: Algebra and number theory	11 years' experience Associate Professor Since 2015, he has been teaching abstract algebra. He wrote articles on algebra and number theory.	Undergraduate level: Abstract algebra, Linear algebra, Basic mathematics, Graph theory and applications, Math applications, Abstract algebra Postgraduate level: Special topics in algebra, Graph theory and applications, Graph matrices and applications

Figure 1. Participants' characteristics

Figure 1 illustrates the selection of instructors with varying academic designations to promote diversity. All participants, except for Instructor2, completed their undergraduate degrees in mathematics and their doctorate degrees in theoretical mathematics. Instructor1 and Instructor3 have instructed courses in teaching mathematics and engaged in research on algebra, number theory, and mathematics education. All participants possess a minimum of 11 years of professional experience and have taught abstract algebra for a considerable duration.

4.3. Data collection tools and process

Semi-structured interviews served as the main data collection method for analyzing instructors' experiences in teaching abstract algebra. The documents supplied by the instructors during the interviews served as additional data sources. The interview protocol, developed in alignment with the research questions, comprises five primary sections: student understanding, curriculum, content knowledge, teaching methods and techniques, and assessment and evaluation (Table 1).

Table 1. The interview protocol

Aim	Questions
Student understanding	· What method do you employ to recognize the individual differences among your students when teaching abstract algebra? Could you provide an example to clarify? · What can you say about your experiences regarding the prerequisites your students should have for the abstract algebra course? · How do you identify potential misconceptions, learning difficulties, or challenges that students may encounter in the abstract algebra course? Please explain with an example. · Please share your experiences concerning mathematical solutions, discussions, explanations, and problem-solving methods in relation to student participation in the abstract algebra course.
Curriculum	· What criteria do you use to select the concepts or topics for teaching abstract algebra? a) How is the course content prepared? b) Do you consider the topics included in the curriculum to be appropriate? What is the reason for this? c) Do you highlight crucial points pertaining to concepts or subjects?
Content knowledge	· How do you prepare to clarify topics or concepts in your abstract algebra class? · What is your methodology for introducing a new concept in an abstract algebra course? · What do you pay attention to when explaining a topic or concept, giving examples, and using symbols in the abstract algebra course? · What factors do you consider when choosing exercises and problems for classroom use? · What factors do you consider when presenting various solutions to the exercises and problems utilized in the class? · Which topics and concepts from the abstract algebra course have real-world applications? · How do you encourage your students to make connections between concepts in abstract algebra? Could you share your experiences? · What methods do you develop to facilitate your students' understanding of the topic or concept in abstract algebra, considering the difficulties they face and the misconceptions they have?

Teaching methods and techniques	· What teaching methods do you employ in the instruction of abstract algebra? (At the beginning of the lesson? In highlighting the topic? In deepening the topic? At the end of the lesson? To ensure that students think and conduct research?) · Have you ever altered your teaching approach for a specific topic? Please provide an explanation. · What concepts or topics do your students find challenging? · What strategies do you employ to address students' challenges with concepts? · What strategies can be employed to address the misconceptions identified in students during the abstract algebra course? · Do you utilize educational technologies (smart boards, computer algebra systems, etc.) in your abstract algebra course? What are your justifications? Can you discuss your experiences?
Assessment and evaluation	· What assessment and evaluation methods are employed in the abstract algebra course? · Which assessment tools do you use? (For identifying errors and misconceptions, encouraging higher-order thinking, determining learning levels, evaluating exam papers) What are your justifications?

Expert feedback was obtained from four faculty members with doctoral degrees in mathematics education and 14 to 22 years of professional experience to assess the relevance and clarity of the interview questions outlined in Table 1 in relation to the research objectives.

In the data collection phase, volunteer instructors were identified, and interviews were conducted in a quiet office setting to ensure their comfort. Before the interviews, instructors were allowed to examine the interview questions, and comprehensive responses were promoted. The interviews commenced with inquiries including, "What is your area of expertise?" and "How many years of professional experience do you possess?" During the interviews, two participants agreed to audio recordings, whereas the other two opted for written notes. Researchers have abstained from directing instructors during the interviews. Instructors facing difficulties were afforded short breaks, and interviews lasted approximately 65 minutes. Images of materials were obtained, and remarks from individuals who declined audio recording were included in the dataset.

4.4. Data analysis

The data analysis involved the consolidation of audio recordings, written notes, transcripts, field notes, and documents acquired from participants. The researchers repeatedly checked the data to guarantee completeness and assessed it within the context of the components of pedagogical content knowledge (Shulman, 1986, 1987) through content analysis. The interview questions (see Table 1) were classified into themes: student understanding, content, teaching methods and techniques, assessment and evaluation, and curriculum knowledge. The data derived from questions pertaining to these themes were analyzed within their respective categories and not employed for other thematic classifications. For instance, data on student understanding provided insights into instructors' content knowledge and assessment practices; however, these components were not integrated with other research questions. The analysis revealed that student understanding was categorized into three themes (Figure 2), content knowledge into nine themes (Figure 3), teaching methods and techniques into four themes (Figure 5), assessment and evaluation into three themes (Figure 6), and curriculum knowledge into four themes (Figure 7).

To guarantee the study's validity and reliability, coding was conducted independently by the researchers, and the consistency among codes was assessed using Miles and Huberman's (1994) reliability coefficient formula [Reliability = Number of Agreements / (Number of Agreements + Number of Disagreements)]. The inter-coder agreement rate was determined to be 94%, with any discrepancies addressed by the participation of a third researcher, resulting in complete consensus. To improve validity, the procedure was meticulously detailed, and the codes and themes were confirmed with participant statements (Maxwell, 1992).

4.5. Ethical Considerations

Instructors were informed that participation in the study was entirely voluntary and that they might withdraw at any time without consequences. The instructors were informed of the study's topic of interest. Researchers guaranteed instructors that confidentiality would be preserved in any written reports derived from the study.

5. Results

5.1. Experience Regarding the Knowledge of Students' Understanding

The instructors underscored the essential significance of students' understanding during the teaching of abstract algebra. They elaborated on this understanding of students' prior knowledge, problem-solving approaches, and learning difficulties (Figure 2).

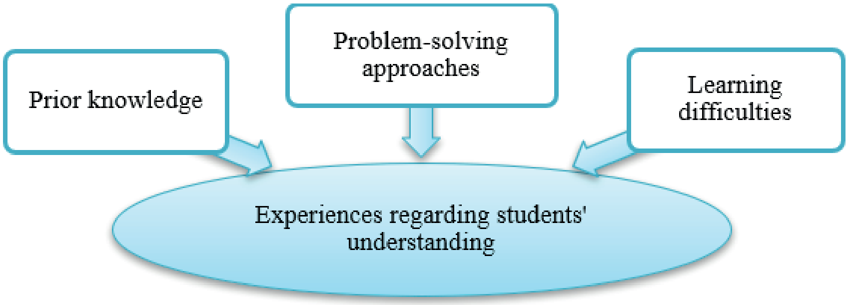

Figure 2. Themes for experiences regarding students' understanding

The instructors highlighted the significance of understanding students and considering prior knowledge in the teaching of abstract algebra. They stated that abstract algebra is founded on abstract mathematics and linear algebra, emphasizing the essential role of understanding number theory, algebraic structures, and operational properties. In instances of inadequate prior knowledge, the process of understanding students was advanced by addressing these gaps. Instructor1 articulated their methodology for assisting students in linking abstract algebra concepts to equation solving, noting, "I help students recognize that they utilize abstract algebra concepts daily, even in solving simple equations such as $2x + 1 = 5$." Instructor2 articulated the importance of evaluating students' prior knowledge at the beginning of each lesson or when presenting new topics, stating, "At the start of every lesson or new topic, I ask them, "What do you know about this topic?" or "How much do you know?"" Instructor3 highlighted the significance of number theory and the need for remedial measures in instances of knowledge deficiencies, asserting, "They need to know number theory; if they don't understand divisibility rules, they can't do algebra. Knowledge of abstract mathematics and proof techniques is also essential. If they don't, I must address those deficiencies." Instructor4 similarly believed that abstract mathematics and linear algebra serve as foundational courses and indicated the implementation of activities in the initial two weeks to remediate prior knowledge deficiencies.

The instructors analyzed students' problem-solving methods to understand them better. To accomplish this, they utilized techniques including having students solve problems on the board, deliberately presenting incorrect solutions to assess students' awareness, and posing standard questions. Instructor2 described their method of monitoring student responses by having them solve problems on board while deliberately triggering errors.

They stated, "I lead the problem-solving in the wrong direction and continue solving it, noting when no one reacts or when they immediately catch the error." Instructor3 noticed that students often exhibit comparable errors and stated, "When I pose questions aimed at emphasizing specific and common mistakes, I implement activities that reflect the clarity of those errors."

In recognition of the challenges posed by abstract algebra concepts for students, instructors strategically structured course content to mitigate these difficulties. Instructor1 noted that students face abstract concepts, including residue classes, quotient groups, and permutation groups, for the first time, which can be challenging to understand. Instructors generally agreed that, although abstract algebra presents challenges, these can be addressed through strategies that minimize repetition and rote memorization. Instructor1 endorsed this perspective by referencing Turkish mathematician Ali Nesin's assertion: "When students read an algebra textbook for the first time, they understand nothing; during the second reading, they grasp some points; and by the third reading, they fully comprehend the topic." Instructor3 indicated that they start with familiar concepts and progressively advance to more complex ones to enhance comprehension, summarizing this method as: "I try to simplify as much as possible. Before addressing binary operations, it is essential to first examine relations. When discussing relations, I begin with concepts familiar to students, such as their preferred functions, and subsequently develop the topic from that foundation. This method demonstrates greater efficacy."

5.2. Experience Regarding Content Knowledge

In teaching abstract algebra, instructors emphasized the necessity of tracking a logical order, forging conceptual links, devising strategies to facilitate comprehension, ensuring clarity in explanations and symbols, employing multiple representations, allowing adequate time, valuing student ideas, relating concepts to real-world contexts, and promoting peer interaction (Figure 3).

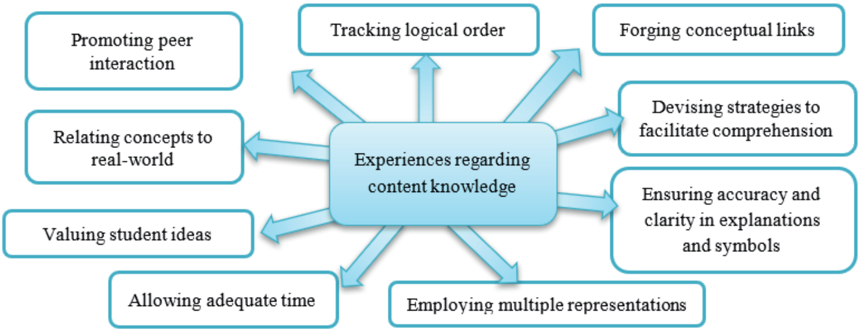

Figure 3. Themes for experiences regarding content knowledge

Instructors typically presented examples in abstract algebra from simple to complex following the introduction of theoretical knowledge. They meticulously provided logical and diverse examples to deepen conceptual understanding. Instructor1 indicated a preference for the sequences "definition-example-theorem" or "definition-theorem-example," while rarely employing the sequence "example-example-definition-theorem." Instructor2 stated a preference for beginning with straightforward examples prior to advancing to more complex ones. Instructor3 emphasized the efficacy of starting with familiar knowledge. Instructor4 highlighted the principle that "the best example is the logical one" and stressed the significance of demonstrating various solution methods.

Instructors highlighted the necessity of maintaining accuracy and clarity in explanations and symbols during content presentation. Their emphasis was on the accurate use of mathematical terminology and the historical context of symbols, demonstrating a zero tolerance for incorrect or incomplete information. They pointed out, nonetheless, that their teachings are not shaped by student preconceptions. In order to highlight the significance of effectively utilizing symbols and their origins, Instructor 1 said, "I emphasize that symbols should be used correctly." In their own words, "I touch upon the origin of the symbols as far as my knowledge goes." Furthermore, they stressed that interpreting the $\otimes$ or $\odot$ symbols only as multiplication causes misunderstandings, although they actually indicate a generic operation. "Algebra is like the links of a chain, it must progress without breaking," they said, emphasizing the need of using symbols correctly. Instructor2 emphasized the necessity of teaching commonly used symbols alongside their correct names to promote mathematical culture and to mitigate conceptual misunderstandings through this approach. Instructor3, emphasizing the proper application of mathematical symbols, expressed his expectation for accurate notation in fundamental concepts and stated that they warned students about their erroneous methodologies. They asserted that mathematics is a universal language and that all should comprehensively understand its written form. Instructor4 asserted that they concentrate exclusively on mathematical terminology in their classes and endeavors to remain within these parameters.

In abstract algebra courses, instructors utilized various representations, including tables, particularly Cayley tables, and graphs, to elucidate topics and concepts. Instructor2 asserted that these representations serve as an effective teaching tool, stating, "In group-related questions, we can evaluate whether a structure is a group by creating a table. A symmetrical table indicates the presence of commutative property; however, this assessment is contingent upon the subject's nature. Similar evaluations can also be conducted using graphs, diagrams, or sets." Instructor3 confirmed that they utilized a table and diagram in Figure 4 to clarify the group properties of algebraic structures including $Z, Q, R, Z^+, Q^+, R^+, Z^*, Q^*, R^*$. Instructor1 and Instructor4 did not provide any information regarding the use of representations.

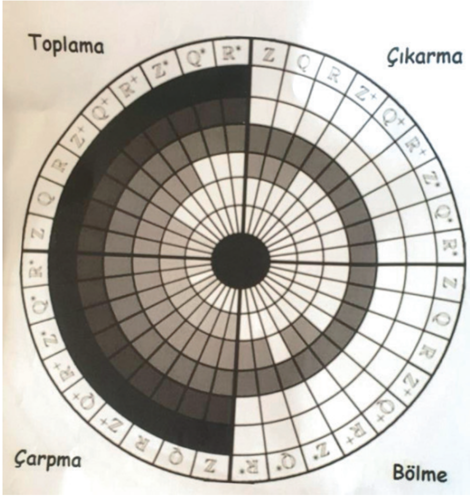

Figure 4 A diagram used by Instructor3 in the abstract algebra course

Instructors noted that although relating concepts to real-world contexts presents difficulties, they incorporate these connections when they would be beneficial. Instructor1 clarified that the associative property of addition is frequently used in daily life, using the operation $2 + 3 + 5$ to show how one may write $2 + (3 + 5)$ or $(2 + 3) + 5$. Instructor2 pointed out that although abstract algebra has few practical uses, real-world scenarios captivate students. Instructor3 clarified via clock arithmetic the use of quotient groups in the teaching of abstract algebra. Conversely, Instructor4 connected advanced concepts including vector spaces, isomorphisms, and dimension to real-world contexts, showing with the example, "Removing vectors 1, 2, and 3 from R^3 space creates a 'black hole' effect, absorbing an infinite number of vectors."

Due to the cumulative nature of abstract algebra, instructors highlighted the interconnectedness of concepts in the abstract algebra, noting that neglecting these relationships may result in misconceptions. They emphasized the significance of meticulously establishing conceptual links at the course's outset. Instructor1 highlighted the significance of these connections, stating, "The concepts in algebra resemble the links of a chain; if one link fails, the entire structure is compromised." Instructor2 elucidated the relationship between the definitions and properties of groups and subgroups, highlighting the significance of inter-concept connections by stating, "We are trying to relate somethings; however, if the student does not fully understand these concepts, they cannot make the transitions." Instructor3 remarked, "If a student misunderstands the equivalence relation or divisibility, they will persist on an incorrect trajectory." emphasizing the enduring consequences of erroneous learning. Instructor4 indicated that he assigns homework to reinforce the relationships among concepts.

Instructor2 and Instructor3 said that students share the topics they struggle with or exam-related discussions with their peers and they encourage such peer interactions within the classroom. While Instructor3 showed through examples that exam-related topics are discussed in class and students share their mistakes with one another, Instructor2 underlined that students seek help from peers regarding areas they are hesitant to ask about.

Instructors prioritized the assessment of students' mathematical comprehension, learning processes, misconceptions, and original problem-solving approaches. Instructor1 indicated that they employ a method where they call the student to the board to solve the problem, allowing them to recognize their mistakes. Instructor2 stated, "Whether in an exam or on the board, I accept any solution that I find logical," highlighting his appreciation for students' mathematical perspectives and their intention to address errors promptly. Instructor3 noted that students frequently discover original solutions and emphasized the importance of rewarding students by sharing such solutions in the classroom, thereby encouraging original problem-solving methods. Instructor4 expressed their support for students' mathematical approaches, stating, "After showing proof to the students, I expect them to continue with the solution."

All instructors indicated that the allocated time is inadequate, as the abstract algebra course meets only three hours per week. Instructor3 noted that, despite time constraints, they provide students with opportunities to engage with questions. They stated, "We have a scheduling issue; however, after writing the question on the board, I walk around the classroom and encourage students to solve it on their own." Instructor1 indicated that they reinforce the material through supplementary assignments at the conclusion of each topic, whereas Instructor2 noted that they either assigns homework or elaborates on the topic based on the students' understanding levels.

The instructors indicated a preference for simplifying topics and employing diverse methods to deal with students' difficulties or misconceptions, as well as to provide various solutions. Instructor2 indicated that they elucidate concepts that are not comprehended through various approaches until student understanding is achieved. Instructor3 highlighted the significance of clear explanation, stating, "Expressing a topic simply demonstrates your understanding; if you cannot simplify it, you are unable to explain it effectively." Instructor4 indicated that they offer multiple examples to enhance student learning.

5.3. Experience Regarding the Knowledge of Teaching Methods and Techniques

The instructors conveyed their experiences in implementing selected methods and techniques to support student learning in abstract algebra. These experiences are classified into categories including the integration of various methods, effective method utilization, student engagement, and equipping students with the ability to mathematize solutions (Figure 5).

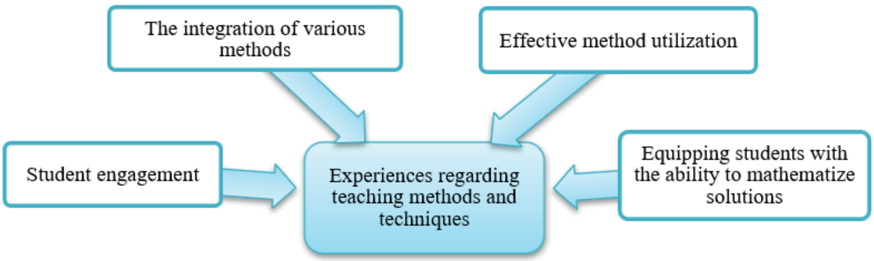

Figure 5. Themes for experiences regarding teaching methods and techniques

In the abstract algebra course, instructors employed various methods and techniques, predominantly utilizing the lecture method. Instructor2 indicated that they integrate lecture method, demonstration, and question-answer techniques based on the topics addressed, stating, "I initiate the topic by presenting an example or solving a problem, and I expect students to solve similar ones." Instructor3 noted that they typically teach lessons in a conversational manner, employing the question-and-answer method, and incorporate diagrams and concept maps to clarify the concepts of groups and subgroups. Instructor4 highlighted the integration of technology in lessons through presentation and question-answer methods, incorporating tasks like writing algorithms and programming on a computer. Conversely, Instructor1 considered technology inappropriate for abstract algebra, asserting, "I use the lecture method... I don't find technology appropriate for abstract algebra course."

Only Instructor3 and Instructor4 offered detailed insights into the effective implementation of the teaching methods and techniques they employed. Instructor3 articulated that they employ concept maps to enhance understanding of conceptual comprehension and the relationships between concepts through the following statements:

"The kernel is a normal subgroup of a group. I constructed a concept map to elucidate the definition of a normal subgroup. This approach allows the student's understanding of the information provided, thus improving their understanding as I transition to proof."

The instructors noted that the teaching environments they created focused on fostering student engagement. They utilized various methods including posing challenging questions, exploring the origins of concepts, clarifying the applications of theorems, and requesting examples. Instructor4 indicated that they frequently posed questions during the class. Instructor1 mentioned, "Sometimes I present interesting or challenging research questions," indicating their efforts to enhance student engagement in research beyond the classroom. Instructor2 highlighted the interconnected of theorems, stating, "I explain in detail the origins of other lemmas or statements within a theorem," which successfully kept students engaged. Instructor3 emphasized that proving theorems alone is inadequate; it is essential to demonstrate their application through examples, as shown in the following statements:

"I provide examples concerning isomorphism theorems. A student familiar with the isomorphism theorem should be able to apply it effectively. While the theorem can be stated and proven by all, challenges may emerge in its application. For

instance, when exploring an isomorphism theorem, a student should understand groups, quotient groups, kernels, and images. This approach helps students identify and fill their knowledge gaps."

The instructors indicated that students inadequately employ diverse mathematical methods during examinations, in-class problem-solving, and theorem proofs. Instructor2 indicated that students presenting original and logical solutions are awarded 5-10 points. Instructor3 indicated that rather than directly solving problems in class, they await students' solutions, resulting in 2-3 distinct approaches, occasionally incorporating methods previously unconsidered by the instructor. Instructor4 emphasized the importance of collaboration, stating, "We find the solution path together through discussion" when proving theorems.

5.4. Experience Regarding the Knowledge of Assessment and Evaluation

Instructors reported that they crafted the assessment and evaluation processes to incorporate questions aimed at identifying student errors and misconceptions, fostering higher-order thinking, and offering feedback on student work (Figure 6).

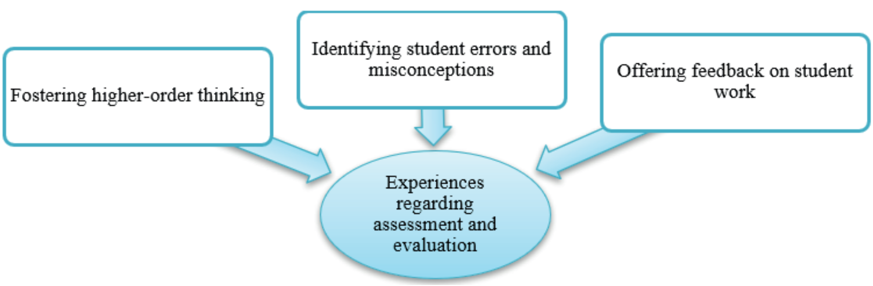

Figure 6. Themes for experiences regarding assessment and evaluation

The instructors used various strategies to identify student errors and misconceptions, such as intentionally presenting incorrect solutions, asking critical questions, discussing examples in class, and employing inquiries that promote deeper conceptual understanding. Instructor1 noted that they illustrate potential mistakes in proof and example solutions, offering explanations for their emergence. Instructor2 promptly addressed students' errors by inviting them to the board, fostering an interactive environment. Instructor3 highlighted their approach to identifying misconceptions through critical questioning, stating, "There are some decisive questions; a student with misconceptions makes mistakes on these questions." Instructor4 emphasized that students often make mistakes with the associative property of matrix multiplication and stressed the importance of working through different examples.

The instructors indicated that they employ strategies including concept reinforcement, assignments, unproven theorems, and challenging questions to promote higher-level thinking among students. Instructor1 asserted the necessity of reinforcing conceptual knowledge by stating, "I pose challenging questions and assigns homework to promote higher-order thinking." Instructor2 articulated their approach by stating that they promote student research through award-winning questions and offer an additional 10 points on the midterm for correct answers. Instructor4 highlighted the significance of unproven theorems and open-ended questions. Instructor3 concentrated on the concepts of groups, rings, fields, and quotient groups, presenting perplexing questions as illustrated below:

"What would occur in the absence of normal subgroup? Why is the normal subgroup necessary when a subgroup already exists? In what circumstance is the normal subgroup used instead of other group structures in the quotient group? or why should we use the ideal in the context of the quotient ring?"

Instructors offered feedback via in-class discussions to clarify exam questions and rectify misconceptions. Instructor1 indicated that they encourage class discussions during lessons and exam evaluations to address misconceptions. Instructor2 indicated that feedback was given regarding common errors and strategies for enhancing performance following the exam. Instructor3 indicated that they addressed students' grade expectations post-exam and communicated the areas where mistakes occurred.

5.5. Experience Regarding the Knowledge of Curriculum

In the course of teaching abstract algebra, instructors conveyed their experiences regarding the curriculum their experiences regarding the curriculum, focusing on themes such as delineating the boundaries of topics and concepts, pinpointing critical points, emphasizing fundamental knowledge and skills, and considering prior topics and concepts (Figure 7).

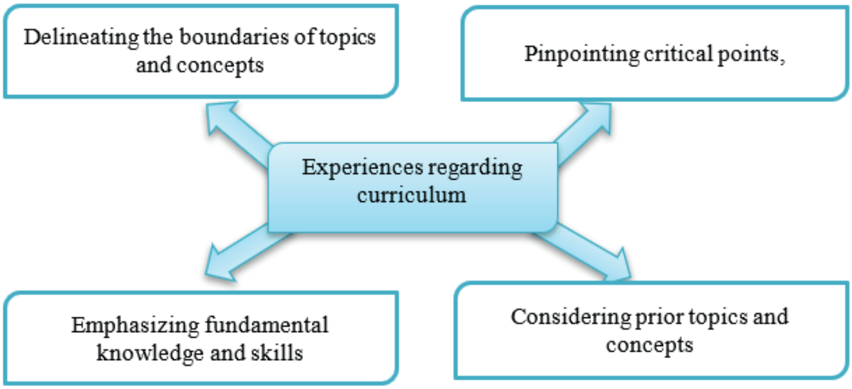

Figure 7. Themes for experiences regarding curriculum

All instructors stated that they delineate the boundaries of topics and concepts based on the CoHE (Council of Higher Education) curriculum and also employ various sources and lecture notes. Instructor1 noted that their teaching was grounded in the CoHE curriculum, utilizing lecture notes and diverse sources due to the variances and shortcomings in the methodologies of these sources. Instructor2 expressed a commitment to providing effective instruction by incorporating investigations from diverse sources into the curriculum. Instructor3 indicated that the CoHE curriculum served as a basis, with the order of topics arranged according to their pedagogical preferences. Instructor4 emphasized the strengthening of course content by integrating both domestic and international resources, alongside the CoHE curriculum.

The instructors indicated that the credits and hours allocated to the abstract algebra course do not adequately fulfill the curriculum requirements. Instructor1 remarked that the curriculum topics cannot be adequately addressed with the existing credit allocation. Similarly, Instructor3 expressed that the 3-hour class duration is insufficient for a thorough coverage of all abstract algebra topics. Instructor4 highlighted the insufficiency of the course credit in relation to the demanding curriculum and indicated that they were evaluating the overall class circumstances.

All instructors reported that they utilized various methods, including making connections, providing counterexamples, emphasizing key points, vocal emphasis, and employing question-answer techniques to underscore essential points in the abstract algebra course. Instructor1 highlighted the relationship between critical points and various mathematical domains, while Instructor4 favored the use of counterexamples for illustration. Instructor2 emphasized critical points to students by utilizing textbooks and elevating their voice for greater emphasis. Instructor3 noted that they conveyed the essential points through the question-and-answer method.

The instructors underscored the significance of integrating previously covered topics and concepts in the curriculum when choosing exercises and problems for the abstract algebra course and stressed the need to establish connections among them. Instructor2 elucidated this situation by stating, "Without knowledge of a group, one cannot form a subgroup and consequently cannot progress to a normal subgroup. Additionally, one cannot define a ring by introducing an alternative multiplication operation, nor can one progress to a field; these concepts are all interrelated. They need to establish a strong connection to stack them sequentially and interlink them."

The instructors emphasized fundamental skills, including concept definitions, hierarchies among concepts, and proof skills, while also evaluating various sources for content presentation. Instructor1 indicated conducting a literature review for theorems and concepts, whereas Instructor2 noted a broad perspective in their approach to the topics. Instructor3 highlighted the necessity of integration, stating, "Book A contains only theorem proofs, while Book B consists solely of examples; we must combine these." Instructor4 emphasized the significance of examining the hierarchical relationships among concepts and their contextual relevance.

6. Discussion and Conclusion

This study offers a focused perspective on the teaching practices of four instructors who teach abstract algebra teaching. The findings underscore the multifaceted aspects of teaching and correspond with the current literature emphasizing the significance of comprehensive teaching strategies (Wasserman, 2017; Zbiek & Heid, 2018). The diverse experiences and educational backgrounds of instructors have resulted in notable variations and conflicting approaches in both content knowledge and pedagogical practices (Manandhar & Sharma, 2021; Suominen, 2018). This highlights the necessity of maintaining consistency in teaching practices and creating effective instructional designs.

Teaching is characterized as a mosaic influenced by various factors, much like a complex artwork formed by each brushstroke of a painter (Fukawa-Connelly et al., 2016; Johnson et al., 2018; Subedi, 2020; Wasserman, 2016). This study demonstrates that the interaction among understanding students, content knowledge, teaching methods, assessment strategies, and curriculum design is essential in abstract algebra instruction. Recognizing students' prior knowledge, misconceptions, and learning difficulties is essential for effective teaching (Gnawali, 2024). The findings demonstrate that instructors employ assessment methods to identify student errors and implement appropriate measures. Courses such as abstract algebra require instructors to implement innovative teaching strategies (Alam & Mohanty, 2024; Gnawali, 2024; Manandhar & Sharma, 2021; Subedi, 2020; Veith et al., 2022a; Veith et al., 2022c). Furthermore, identifying student errors and leveraging them to enhance teaching reinforces assessment and feedback mechanisms (Alam & Mohanty, 2024; Gnawali, 2024; Tanışlı, 2013; Veith et al., 2022b; Veith et al., 2022c). Nevertheless, the findings indicate that instructors frequently prioritize content knowledge and assessment, neglecting to leverage student errors as a means to improve the teaching process. This contrasts with literature indicating that student errors can influence teaching practices (Booth et al., 2013; Metcalfe et al., 2024).

Assessment and evaluation practices play a critical role in improving the effectiveness of abstract algebra instruction and enriching student learning. These practices not only assess student performance but also contribute to the development of instructors' pedagogical strategies (Durkin et al., 2021; Fortes, 2016; Litke, 2019; Veith et al., 2022a; Veith et al., 2022b). Multiple assessment methods serve to link curriculum goals and teaching methods, functioning to deepen conceptual understanding and address misconceptions (Capaldi, 2014; Soto-Johnson et al., 2009). Instructors' efforts to guide students towards higher-order thinking are manifested through thoughtfully designed tasks, open-ended problems, and interactive discussions. Methods including the discussion of exam questions and an emphasis on learning outcomes facilitate cognitive development in students (Dubinsky & Leron, 1994). Conversely, the absence of standardized rubrics complicates the achievement of consistency in evaluation processes (Alam & Mohanty, 2024; Gnawali, 2024; Litke, 2019; Wheeler & Champion, 2013). The research indicated that instructors employ inquiry-oriented instruction via assignments, open-ended questions, and challenging problems to improve student interest in abstract algebra topics (Capaldi, 2014; Haider & Andrews-Larson, 2022; Khasawneh et al., 2023). The restricted application of these methods underscores the necessity for more extensive strategies. Innovative approaches, including Melhuish's (2019) Group Theory Concept Assessment (GTCA) and Soto et al.'s (2024) suggestion to combine assessment processes with embodied activities, present opportunities to improve student learning. The traditional grading system is believed to restrict student engagement, while dynamic and interactive methods may enhance teaching quality significantly. The integration of computer software such as ISETL, GAP, MAPLE, and MAGMA in abstract algebra courses offers significant potential for improving student outcomes and diversifying instructional methods (Krishnamani & Kimmins, 2001; Mrope, 2024; Nwabueze, 2004; Okur et al., 2011).

Shaping content knowledge based on the structure and boundaries of the curriculum is a widely accepted principle in pedagogical literature (Grootenboer et al., 2023). In abstract algebra instruction, content knowledge is a crucial factor that directly influences students' comprehension. Instructors emphasized that the abstract algebra course builds upon previous mathematics courses and necessitates a more extensive curriculum than what CoHE recommends. Instructors prioritized the cumulative structure of the course, conceptual connections, and proof skills, while also addressing deficiencies through supplementary materials. This approach aligns with the recommendations of Wasserman (2016) and Gnawali (2024) on improved curriculum design. The embodied activity proposals by Soto et al. (2024) offer the potential to facilitate transitions between topics and to mitigate instructional challenges. The intensity of abstract algebra course content and time constraints are frequently identified as common issues in literature (Gnawali, 2024; Grassl & Mingus, 2007; Leron & Dubinsky, 1995; Subedi, 2020). This situation hinders the capacity to address questions and concentrate on students' needs (Clark et al., 1997; Fukawa-Connelly et al., 2016). Nevertheless, tools like diagnostic questions, Cayley tables, and graphs have helped students grasp complex concepts (Findell, 2001; Manandhar & Sharma, 2021). These findings correspond with Gnawali's (2024) suggestions for addressing formalism challenges and the efforts of Soto et al. (2024) to reduce abstraction. In summary, the complex connections between curriculum and content knowledge deserve deeper exploration, and creating innovative pedagogical solutions is crucial to tackling the challenges faced by instructors. The effective teaching of abstract algebra courses relies on instructors implementing student-centered pedagogical strategies. Although the course content is complex and intensive, various methods are employed to enhance students' conceptual understanding. Among these methods, as highlighted in the literature (Fukawa-Connelly, 2012), are question-answer interactions, concept inquiry, an emphasis on the applications of theorems, and the promotion of diversity in solution methods. It is recognized that lessons typically rely on direct instruction, and there is inadequate support for student participation. This contrasts with findings in the literature that support the effectiveness of constructivist techniques (Capaldi, 2014; Clark et al., 1999; Dubinsky & Leron, 1994; Fukawa-Connelly et al., 2016).

The uncommon preference for tools like concept maps and visual materials suggests that their usage is restricted. This finding challenges the conclusion drawn by Johnson et al. (2018), which indicates that instructors tend to favor out-of-class teaching methods due to constraints imposed by their beliefs and contextual factors. Furthermore, while instructors hold differing opinions regarding the incorporation of technology, existing literature highlights that software designed for abstract algebra enhances the comprehension of concepts (Krishnamani & Kimmins, 2001; Mrope, 2024; Nwabueze, 2004; Schubert et al., 2013). Software such as ISETL (Krishnamani & Kimmins, 2001), semiotic approaches (Findell, 2001), and tools like GTCA (Melhuish, 2019) serve as effective methods to enhance relational understanding in the teaching of abstract algebra. Wasserman (2017) underscored the necessity of these methods by pointing out the significance of conceptual connections. Moreover, it has been observed that representations like the easy-to-hard learning sequence, Cayley tables, and operation tables, which reinforce theoretical knowledge through straightforward examples, effectively enhance conceptual understanding. Research indicates that multicolored Cayley tables are effective tools for teaching group theory. By using concrete examples and visual metaphors, instructors can enhance students' comprehension (Findell, 2001; Manandhar & Sharma, 2021; Nwabueze, 2004). Furthermore, the visualization proposal by Schubert and colleagues (2013) acts as a valuable guide for deepening learning processes. While connecting abstract algebra concepts to real-world applications is essential for enhancing understanding, instructors have noted challenges in making these connections. Methods like simulations and gestures enhance the engagement and comprehension of abstract concepts (Soto et al., 2024). In summary, approaches that incorporate diverse teaching strategies, utilize materials effectively, and create connections to real-world contexts have proven effective in the instruction of abstract algebra (Agustyaningrum et al., 2021; Gnawali, 2024).

This research helps to clarify the relevance of pedagogical content knowledge in teaching abstract algebra. The findings suggest that instructors underevaluate the possibilities of utilizing student mistakes as an active learning mechanism, and that innovative assessment methods remain restricted. The findings reflect existing literature and offer a holistic view of improving abstract algebra instruction as well as some significant implications for future studies.

7. Limitations and Suggestions

This study focuses on the instructional experiences of instructors in abstract algebra courses, rather than concentrating on particular mathematical subjects. Future research may concentrate on specific areas including groups, rings, fields, normal subgroups, and isomorphism. This research is limited to four instructors possessing doctoral-level expertise in algebra and number theory. Comprehensive analyses may be enhanced through extensive studies that incorporate instructors from diverse educational backgrounds.

This research utilized interviews to collect participants' experiences and perspectives. The absence of classroom observations has constrained the depth of the findings obtained. Incorporating both interviews and observations into a more comprehensive research design can effectively address this limitation. This research employs a qualitative approach, yet future research could adopt experimental designs to evaluate the effectiveness of specific teaching methods in abstract algebra.

Research shows that instructors have diverse perspectives on technology use, highlighting a need for further studies on its impact in teaching abstract algebra. Furthermore, it has been observed that instructors limit student interactions primarily to the teacher-student dynamic, often preferring to facilitate student-student interactions outside the classroom settings. This situation could have an indirect effect on student success by limiting active classroom participation. This finding leads to three recommendations: 1) Encouraging instructors to create environments that support active participation, 2) Examining how active participation influences students' cognitive and affective outcomes, 3) Performing in-depth analyses of teaching practices that foster active participation.

This study noted a limited use of alternative assessment methods. Instructors are advised to implement alternative assessment methods, including portfolios. Researchers (Capaldi, 2014; Fortes, 2016; Litke, 2019; Soto-Johnson et al., 2009) have indicated that portfolios enhance the effective use of mathematical language and support individual student development.

Declarations

Author contributions: [Fatma Sümeyye Uçak] contributed to the conception, design, and execution of the study, including data collection, analysis, and interpretation. [Tuğba Horzum] provided guidance and oversight throughout the research process. The manuscript was drafted by [Fatma Sümeyye Uçak and Tuğba Horzum] and revised collaboratively by all authors.

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