Chapter 1

Neimark-Sacker Bifurcation in a Nonstandart Finite Difference Discretized Financial System

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Abstract

Neimark-Sacker bifurcation theory is used to analyze a differential equation system. A three dimensionel continuous time system is investigated here. Generally, forward Euler scheme is used for discretization from continuous time system to discrete time system. But, in this study Nonstandard Finite Difference (NSFD) scheme is applied to discrete the continuous time system to discrete time system. After that, we examine topological structure of equation. Then, we give the conditions of local stability of this system around feasible fixed point. After, we show analitically that discretized system undergoes Neimark-Sacker bifurcation when one of the system parameter varies near its critical value. We confirm the existence of Neimark-Sacker bifurcation via explicit Flip and Neimark-Sacker bifurcation criterion. And we determine the direction of bifurcation with the help of center monifold theory and bifurcation theory. We carry out numerical simulation to approve our analitical findings.

1. Introduction

Systems of differential equations play an important role in understanding almost all phenomena in nature. These equations are used to explain the relationships between variables, whether past or future. Furthermore, solving these equations is crucial for predicting similar future events or interpreting past events. There are two types of the mathematical models in the theory of dynamic models: the continuous-time models governed by differential equations, and the discrete-time models described by difference equations. These systems are used in fields such as biology, medicine, economics, chemistry, etc. Economic Dynamics have recently become more prominent in mainstream economics. The financial and economic systems show a lot of complex dynamical phenomena, such as, business cycle, financial crisis, irregular growth. Many dynamics models of economics and

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finance present various complex dynamical behaviors such as chaos, fractals and bifurcation[20].

Bifurcation theory is a powerful mathematical framework that examines changes in the behavior of equations or systems of equations as parameters are varied. It began to take shape in the mid-20th century with contributions from various mathematicians and physicists. Bifurcation theory has wide-ranging applications across numerous scientific disciplines, including biology, ecology, physics, chemistry, economics, and engineering([3], [4], [5], [6], [8], [9], [12], [14], [18], [20], [23], [24]). For example, in biology and ecology, bifurcation theory is used to explain and model population dynamics, ecosystem stability, and the complex interdependencies in biological systems. By understanding the bifurcation mechanisms underlying ecological transitions, researchers can better manage ecosystems and mitigate the effects of environmental changes on species. Furthermore, the course of an epidemic disease will change with the development of a vaccine. Estimating this parameter plays a crucial role in determining the lasting effects of the disease on species and the impact on species populations. For a financial or economic systems there can be disequilibrium thresholds where society decides it cannot afford the increasing cost of misallocated resources as disequilibrium increases. Such a threshold then forces a restructing of the market system. This concept of restructuring to maintain the survival of the system is known as bifurcation theory. A bifurcation in a financial or economic system is a point(or threshold) where the system is restructured to operate at a more acceptable or stable level of disequilibrium. Bifurcations do not usually lead to equilibrium conditions, only to a stable or comfortable disequilibrium condition under which the system can continue to survive [20].

The Nonstandard Finite Differance Scheme[15] is useful method to solve the differantial equations and system. In recent years, the method has been frequently used, particularly in studies aimed at determining the stability of systems([2], [10], [11], [15], [16], [17]) In this study, the method is used for discretize the equations system from continuous time to discretize time system.

In this paper, a nonlinear financial model is examined proposed by Huang and Li[7] as follows:

$$\begin{aligned} \dot{x} &= z + (y - a)x, \\ \dot{y} &= 1 - by - x^2 \\ \dot{z} &= -x - cz \end{aligned} \tag{1}$$

Where x denotes the interest rate(can be negative or positive), y denotes the investment demand, z denotes the price index, a is the saving amount, b is the cost per investment, c is the demand elasticity of commercial markets, and all three constants $a, b, c \geq 0$.

In literature, there are many studies related to this model regarding its solution, stability, branching, etc. [20]. Unlike other studies in this work, the Nonstandard Finite Differance Scheme[15] developed by Michen is applied for the discretization of the system (1) in order to obtain a discrete-time financial system as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\varphi a}(x + \varphi xy + \varphi z) \\ \frac{1}{1+\varphi b}\left(y + \varphi(1 - x^2)\right) \\ \frac{1}{1+\varphi c}(-cz - x) \end{pmatrix} = \begin{pmatrix} e_1(x, y, z) \\ e_2(x, y, z) \\ e_3(x, y, z) \end{pmatrix} \tag{2}$$

where φ is the step size.

2. Local Stability Analysis of Fixed Point

The fixed points of the system (2) are the solutions of the following non-linear equations:

$$\begin{aligned} x &= e_1(x, y, z) \\ y &= e_2(x, y, z) \\ z &= e_3(x, y, z) \end{aligned} \tag{3}$$

By some algebraic computation, we obtain the following lemma.

Lemma 1: **(i)** If $c - b - abc \leq 0$, system (2) has only one fixed point $E_0\left(0, \frac{1}{b}, 0\right)$.

(ii) If $c - b - abc > 0$, system (2) has three fixed points $E_1\left(0, \frac{1}{b}, 0\right), E_{2,3} = \left(\pm\sqrt{\frac{c-b-abc}{c}}, \frac{1+ac}{c}, \mp\frac{1}{c}\sqrt{\frac{c-b-abc}{c}}\right)$.

Given at any fixed point $E(x, y, z)$, the jacobian matrix of the system (2) and the characteristic equation are as follows

$$J(E) = \begin{pmatrix} \frac{\varphi y+1}{\varphi a+1} & \frac{\varphi x}{\varphi a+1} & \frac{\varphi}{\varphi a+1} \\ -\frac{2\varphi x}{\varphi b+1} & \frac{1}{\varphi b+1} & 0 \\ -\frac{\varphi}{\varphi c+1} & 0 & \frac{1}{\varphi c+1} \end{pmatrix} \tag{4}$$

and

$$P(\mu) := \mu^3 + \vartheta_2\mu^2 + \vartheta_1\mu + \vartheta_0 = 0 \tag{5}$$

where

$$\begin{gathered} \vartheta_2 = -tr(J), \\ \vartheta_1 = \begin{vmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{vmatrix} + \begin{vmatrix} j_{22} & j_{23} \\ j_{32} & j_{33} \end{vmatrix} + \begin{vmatrix} j_{11} & j_{13} \\ j_{31} & j_{33} \end{vmatrix}, \\ \vartheta_0 = -|J|. \end{gathered} \tag{6}$$

Firstly, the following lemma concerning the necessary and sufficient criteria for stability around fixed point of system (2) in order to understand the nature of the system around fixed point $E(x, y, z)$ is provided.

Lemma 2: Suppose that $\vartheta_2, \vartheta_1, \vartheta_0 \in \mathbb{R}$. Then, the necessary and sufficient conditions for all roots of μ of the equation

$$\mu^3 + \vartheta_2\mu^2 + \vartheta_1\mu + \vartheta_0 = 0$$

to satisfy $|\mu| < 1$ are

$|\vartheta_2 + \vartheta_0| < 1 + \vartheta_1, |\vartheta_2 - 3\vartheta_0| < 3 - \vartheta_1$ and $\vartheta_0^2 + \vartheta_1 - \vartheta_0\vartheta_2 < 1$([1]).

Now, we examine the local Dynamics of system (2) around fixed points E_0 according the Lemma 2. At E_0 the jacobian matrix is

$$J(E_0) = \begin{pmatrix} \dfrac{b+\varphi}{b(\varphi a+1)} & 0 & \dfrac{\varphi}{\varphi a+1} \\ 0 & \dfrac{1}{\varphi b+1} & 0 \\ -\dfrac{\varphi}{\varphi c+1} & 0 & \dfrac{1}{\varphi c+1} \end{pmatrix}.$$

The characteristic equation of matrix $J(E_0)$ is

$$P := \mu^3 + \kappa_2\mu^2 + \kappa_1\mu + \kappa_0 \tag{7}$$

where

$$\begin{aligned} \kappa_2 &= -\left(\frac{ab^2\varphi^2 + abc\varphi^2 + b^2c\varphi^2 + bc\varphi^3 + 2ab\varphi + 2b^2\varphi + 2bc\varphi + b\varphi^2 + c\varphi^2 + 3b + \varphi}{(ab\varphi + b)(b\varphi + 1)(c\varphi + 1)}\right) \\ \kappa_1 &= \frac{b^2\varphi^3 + ab\varphi + b^2\varphi + bc\varphi + 2b\varphi^2 + c\varphi^2 + 3b + 2\varphi}{(ab\varphi + b)(b\varphi + 1)(c\varphi + 1)} \\ \kappa_0 &= -\left(\frac{b\varphi^2 + b + \varphi}{(ab\varphi + b)(b\varphi + 1)(c\varphi + 1)}\right) \end{aligned} \tag{8}$$

The jacobian matrix $J(E_0)$ have eigenvalues $\mu_1 = \frac{1}{\varphi b+1}, \mu_{2,3} = \frac{1}{2}\left[B \mp \sqrt{B^2 - 4C}\right]$ where $\mu_{2,3}$ satisfy the equation $\mu^2 - B\mu + C = 0$, where $B = \frac{2b+\varphi+\varphi bc+\varphi^2 c+\varphi ab}{(b+\varphi ab)(1+\varphi c)}, C = \frac{b+\varphi+\varphi^2}{(b+\varphi ab)(1+\varphi c)}$.

We obtain the topological classification of E_0 presented in the following lemma.

Lemma 3: If $abc - c + b > 0$, the fixed point is a

(i) sink if $a > \frac{1}{b}$ and $\varphi < min\left\{-\frac{bc+2ab+3+\sqrt{(bc+2ab+3)^2-16(abc+c+1)b}}{2(abc+b+c)}, \frac{1-ab-bc}{abc-b}\right\}$ or $a < \frac{1}{b}$ and $\varphi > max\left\{\frac{-(bc+2ab+3)+\sqrt{(bc+2ab+3)^2-16(abc+c+1)b}}{2(abc+b+c)}, \frac{1-ab-bc}{abc-b}\right\}$.

(ii) sourche if $\varphi > max\left\{\frac{-(bc+2ab+3)+\sqrt{(bc+2ab+3)^2-16(abc+c+1)b}}{2(abc+b+c)}, \frac{1-ab-bc}{abc-b}\right\}$ and it is locally unstable.

(iii) saddle if $-\frac{bc+2ab+3+\sqrt{(bc+2ab+3)^2-16(abc+c+1)b}}{2(abc+b+c)} < \varphi < \frac{-(bc+2ab+3)+\sqrt{(bc+2ab+3)^2-16(abc+c+1)b}}{2(abc+b+c)}$.

(iv) Non-hyperbolic $\varphi = \frac{1-ab-bc}{abc-b}$ or $\varphi = \frac{-(bc+2ab+3)+\sqrt{(bc+2ab+3)^2-16(abc+c+1)b}}{2(abc+b+c)}$.

3. Analysis of Neimark-Sacker Bifurcation

In this section, we focus to discuss the existence, direction and stability analysis of Neimark-Sacker bifurcation of system (2) around fixed points E_0 by using expilicit Flip and Neimark-Sacker bifurcation criterion, Kuznetsov's normal form method and center monifold theory[10]. We take φ as bifurcation theory.

3.1. Neimark-Sacker Bifurcation:Existence, Direction and Stability

3.1.1. Existence of Neimark-Sacker Bifurcation

We will use the expilicit Flip and Neimark-Sacker bifurcation criterion fort he existence of Neimark-Sacker bifurcation as follow lemma([13],[15]).

Lemma 4: Consider n-dimensional discrete system as follows:

$$Y_{k+1} = H_\vartheta(Y_k)$$

where $\vartheta \in \mathbb{R}$ is being taken as a bifurcation parameter. Furthermore, we write the equation of the jacobian matrix $J(Y^*) = (\theta_{ij})_{n\times n}$ at fixed point $Y^* \epsilon\, \mathbb{R}^n$ for H_ϑ as follows

$$D_\vartheta(\mu) = \mu^n + \iota_1\mu^{n-1} + \cdots + \iota_{n-1}\mu + \iota_n = 0 \tag{8}$$

where$\iota_i = \iota_i(\vartheta, \upsilon), i = 1,2,\dots,n$ and υ is being taken as the control parameter unless stated which is to be determined.We define a sequence of determinants $\left(N_i^{\pm}(\vartheta, \upsilon)\right)_{i=0}^{n}$ with $N_0^{\pm}(\vartheta, \upsilon) = 1$ which is to be defined as

$$N_i^{\pm} = det(R_1 \pm R_2) \tag{9}$$

where

$$R_1 = \begin{pmatrix} 1 & \iota_1 & \iota_2 & \cdots & \iota_{i-1} \\ 0 & 1 & \iota_1 & \cdots & \iota_{i-2} \\ 0 & 0 & 1 & \cdots & \iota_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{10}$$

$$R_2 = \begin{pmatrix} \iota_{n-i+1} & \iota_{n-i+2} & \cdots & \iota_{n-1} & \iota_n \\ \iota_{n-i+2} & \iota_{n-i+3} & \cdots & \iota_n & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \iota_{n-1} & \iota_n & \cdots & 0 & 0 \\ \iota_n & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{11}$$

Moreover, we assume that the following conditions hold well:

(i) Eigenvalue Condition: $N_{n-1}^{-}(\vartheta_0, \upsilon) = 0,\ N_{n-1}^{+}(\vartheta_0, \upsilon) > 0,\ D_{\vartheta_0}(1) > 1,\ (-1)^n$

$D_{\vartheta_0}(-1) > 0, N_i^{\pm}(\vartheta_0, \upsilon) > 0$ for $i = n-3,\ n-5, \dots, 2(or\ 1)$ when n is odd(or even respectively).

(ii) Transversality Condition: $\left(\frac{d}{d\upsilon}\left(N_{n-1}^{-}(\vartheta, \upsilon)\right)\right)_{\upsilon=\upsilon_0} \neq 0,$

(iii) Non-resonance Condition: $cos\left(\frac{2\pi}{\ell}\right) \neq \delta$, where $\ell = 3, 4, 5, \dots$ and $\delta = 1 - {0.5\, D_{\vartheta_0}(1) N_{n-3}^{-}(\vartheta_0, \upsilon)} \Big/ {N_{n-2}^{+}(\vartheta_0, \upsilon)}$

then the Neimark-Sacker bifurcation will occur at the critical value ϑ_0.

If we choose $n = 3$, the following lemma will give the necessary and sufficient parametric condition for system (2) underlies Neimark-Sacker bifurcation if bifurcation parameter passes its citical value.

Lemma 5: The Neimark-Sacker bifurcation of system (2) occurs around the fixed point E_0 at $\varphi = \varphi_{NS}$ is and only if

$$\begin{aligned} &1 - \kappa_1 + \kappa_0(\kappa_2 - \kappa_0) = 0 \\ &1 + \kappa_1 - \kappa_0(\kappa_2 + \kappa_0) > 0 \\ &1 + \kappa_2 + \kappa_1 + \kappa_0 > 0 \\ &1 - \kappa_2 + \kappa_1 - \kappa_0 > 0 \end{aligned}$$

$$\frac{d}{d\varphi}\left(1-\kappa_1+\kappa_0(\kappa_2-\kappa_0)\right)_{\varphi=\varphi_{NS}}\neq 0$$
$$cos\left(\frac{2\pi}{\ell}\right)\neq 1-\frac{1+\kappa_2+\kappa_1+\kappa_0}{2(1+\kappa_0)}, \ell=3,4,5,\ldots$$

where $\kappa_2, \kappa_1, \kappa_0$ are given as in (8).

Set

$$NSB_{E_2}=\{(a,b,c,\varphi): \varphi=\varphi_{NS}, a,b,c>0\ \}$$

and for parameter perturbation in a small neighborhood of NSB_{E_0} two roots of (7) equation are comlex conjugate having modules one and magnitude of other root is not equal to one, then the system (2) experiences Neimark-Sacker bifurcation around E_0.

3.1.2. Direction and Stability of Neimark-Sacker Bifurcation

In this section with the help of center monifold theory and bifurcation theory we will determine the direction and stability of the Neimark-Sacker biurcation in system (2). We think the fixed point $E_2=\left(0,\frac{1}{b},0\right)$ of system (2) with arbitrary parameter $(a,b,c,\varphi)\epsilon NSB_{E_0}$. Let $\varphi=\varphi_{NS}$, then the matrix $J(E_0)$ has the eigenvalues satisfying

$$|\mu_i(\varphi_{NS})|=1, i=1,2 \tag{12}$$

and $\mu_3(\varphi_{NS})\neq 1$.

Next, we use the transformation $\hat{x}=x-x^*, \hat{y}=y-y^*, \hat{z}=z-z^*$, where $x^*=0$, $y^*=\frac{1}{b}, z^*=0$ to transfer the fixed point E_0 of system (2) to the origin. After applying Taylor expansion, the system (2) becomes

$$X\longrightarrow A(\varphi)X+F \tag{13}$$

where $A(\varphi)=J(E_0)$ and $X=(\hat{x},\hat{y},\hat{z})^T$ and

$$\begin{aligned} F(\hat{x},\hat{y},\hat{z},\varphi) &= \left(F_1(\hat{x},\hat{y},\hat{z},\varphi), F_2(\hat{x},\hat{y},\hat{z},\varphi), F_3(\hat{x},\hat{y},\hat{z},\varphi)\right)^T \\ &= \left(\frac{\varphi\hat{x}\hat{y}}{1+\varphi a}, -\frac{\varphi\hat{x}\hat{y}}{1+\varphi a}, 0\right). \end{aligned} \tag{14}$$

The system (13) can be written as

$$X_{n+1}=AX_n+\frac{1}{2}B(X_n,X_n)+\frac{1}{6}C(X_n,X_n,X_n)+O({X_n}^4)$$

where

$$B(x,y) = \begin{pmatrix} B_1(x,y) \\ B_2(x,y) \\ B_3(x,y) \end{pmatrix}, \; C(x,y,z) = \begin{pmatrix} C_1(x,y,z) \\ C_2(x,y,z) \\ C_3(x,y,z) \end{pmatrix} \tag{15}$$

are the symmetric multi-linear functions $B: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $C: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ are defined by respectively

$$B_i(x,y) = \sum_{j,k=1}^{3} \left. \frac{\partial^2 X_i(\nu,0)}{\partial \nu_j \partial \nu_k} \right|_{\nu=0} x_j y_k, \quad i = 1,2,3$$

$$C_i(x,y,z) = \sum_{j,k,l=1}^{3} \left. \frac{\partial^2 X_i(\nu,0)}{\partial \nu_j \partial \nu_k \partial \nu_l} \right|_{\nu=0} x_j y_k u_l, \quad i = 1,2,3$$

For the system (12)

$$B(x,y) = \begin{pmatrix} \frac{\varphi x_1 y_2 + \varphi x_2 y_1}{1+\varphi a} \\ \frac{-\varphi x_1 y_1 - \varphi x_2 y_2}{1+\varphi b} \\ 0 \end{pmatrix}, \; C(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{16}$$

For eigenvalues $\mu_1(\varphi_{NS})$ and $\mu_2(\varphi_{NS})$, let $m_1, m_2 \epsilon \mathbb{C}^3$ be two eigenvectors of $A(\varphi_{NS})$ and $A^T(\varphi_{NS})$ respectively satisfying such that the following conditions hold:

$$A(\varphi_{NS})m_1 = \mu_1(\varphi_{NS})m_1, \qquad A(\varphi_{NS})\overline{m_1} = \mu_2(\varphi_{NS})\overline{m_1},$$

$$A^T(\varphi_{NS})m_2 = \mu_2(\varphi_{NS})m_2, \; A^T(\varphi_{NS})\overline{m_2} = \mu_1(\varphi_{NS})\overline{m_2} \tag{17}$$

$$\langle m_1, m_2 \rangle = \sum_{i=1}^{3} m_{1i}.\overline{m_{2\iota}} = 1$$

We separete $X \epsilon \mathbb{R}^3$ as $X = z m_1 + \bar{z}\,\overline{m_1}$ by considering φ vary near to φ_{NS} and for $z \in \mathbb{C}$. The expilicit formula of z is $z = \langle m_2, X \rangle$. So the system (13) transformed to the following system for $|\varphi|$ close to φ_{NS}.

$$z \mapsto \mu(\varphi)z + \hat{g}_{kl}(z, \bar{z}, \varphi)$$

where $\mu(\varphi) = (1 + \hat{\psi}(\varphi))e^{i\theta(\varphi)}$ with $\hat{\psi}(\varphi) = 0$ and $\hat{g}(z, \bar{z}, \varphi)$ is a smooth complex-valued function. Applaying Taylor expansion to the function $\hat{g}$, we obtain

$$\hat{g}(z, \bar{z}, \varphi) = \sum_{k+l \geq 2} \hat{g}_{kl}(\varphi) z^{k-l}$$

with $\hat{g}_{kl} \in \mathbb{C}, \quad k, l = 0,1, \ldots.$

By using symmetric multi-linear vector functions, the Taylor coefficients can be defined

$$\begin{aligned} \hat{g}_{20}(\varphi_{NS}) &= \langle m_2, B(m_1, m_1)\rangle \\ \hat{g}_{11}(\varphi_{NS}) &= \langle m_2, B(m_1, \overline{m_1})\rangle \\ \hat{g}_{02}(\varphi_{NS}) &= \langle m_2, B(\overline{m_1}, \overline{m_1})\rangle \\ \widehat{g}_{21}(\varphi_{NS}) &= \langle m_2, C(m_1, m_1, \overline{m_1})\rangle \end{aligned} \quad (17)$$

The sign of first Lyapunov coefficient $\ell_1(\varphi_{NS})$ determines the direction of Neimark-Sacker bifurcation and is defined by

$$\ell_1(\varphi_{NS}) = Re\left(\frac{\mu_2\hat{g}_{21}}{2}\right) - Re\left(\frac{(1-2\mu_1)\mu_2^2}{2(1-\mu_1)}\hat{g}_{02}\hat{g}_{11}\right) - \frac{1}{2}|\hat{g}_{11}|^2 - \frac{1}{4}|\hat{g}_{02}|^2 \quad (18)$$

where μ_1, μ_2 are pair of complex conjugate eigenvalues, which has been stated in the following theorem.

Theorem: Let suppose (12) holds and $\ell_1(\varphi_{NS}) \neq 0$, then Neimark-Sacker bifurcation at fixed point $E_2(x^*, y^*, z^*)$ for system (2) if the φ changes its value in small neighbourhood of NSB_{E_0}. Moreover if $\ell_1(\varphi_{NS}) < 0$(resp. $\ell_1(\varphi_{NS}) > 0$) then there exist attracting(resp. pepelling) smooth closed invariant curve bifurcate from E_0 and the bifurcation is sub-citical(resp. super-critical)

4. Numerical Simulations

In this section, we will give an example to illustrate theoritical result of system (2) by using numerical simulations with the help of diagrams of bifurcation and phase portraits.

Example: We take $a = 1.8, b = 0.5, c = 0.35$ and find the fixed point of the system (2) $E_0(0, 0.2, 0)$. Then, the bifurcation point is obtained $\varphi_{NS} =$ 0,4054054054 .

The jacobian matrix is evaluated at E_0 is

$$A(\varphi_{NS}) = \begin{pmatrix} 1.046875 & 0 & 0.234375 \\ 0 & 0.831460674 & 0 \\ -0.3550295858 & 0 & 0.8757396449 \end{pmatrix}$$

and the eigenvalues of $A(\varphi_{NS})$ are $\mu_{1,2} = 0.96130732245 \mp 0.275478187395253i$, $\mu_3 = 0.831460674$ with $|\mu_{1,2}| = 1$. Furhermore

$$1 - \kappa_1 + \kappa_0(\kappa_2 + \kappa_0) = 0$$

$$1 + \kappa_1 - \kappa_0(\kappa_2 + \kappa_0) = 0.617346297 > 0$$

$$1+\kappa_2+\kappa_1+\kappa_0=0.0130424748$$

$$1-\kappa_2+\kappa_1-\kappa_0=7.184114459$$

$$1+\kappa_0=0.1685393268$$

$$1-\kappa_0=1.831460673$$

$$\frac{\partial}{\partial\varphi}\big(1-\kappa_1+\kappa_0(\kappa_2-\kappa_0)\big)\neq 0$$

and

$$1-\frac{1+\kappa_2+\kappa_1+\kappa_0}{21+\kappa_0}=0.961307325.$$

From the resonance condition $cos\left(\frac{2\pi}{l}\right)=0.961307325$, we get $l=\pm 22{,}51334970$.

So, the criterion for the existence of Neimark-.Sacker bifurcationare fullfilled with $(a,b,c,\varphi)\in NSB_{E_2}$according to the Lemma 6. Therefore, a neimark-Sacker bifurcation occurs around fixed point E_2 if φ crosses its critical value φ_{NS}.

Let $m_1,m_2\epsilon\mathbb{C}^3$ be two eigenvector of $A(\varphi_{NS})$ and $A^T(\varphi_{NS})$ corresponding to $\mu_{1,2}$ respectively. By some algebraic calculation m_1,m_2 can be found as follows

$m_1\sim(-0.187055600885737+0.602210312930i,0{,}0.77611400011678)^T$, and $m_2\sim(0.77611400011678{,}0{,}0.187055600885737+0.60221031293063)^T$. For

$$<m_1,m_2>=1$$

We can take normalized vector as $m_1=\alpha m_2$, where $\alpha=0.8143679532-1.43354051i$. Then also by (17) and (18) we find $\ell_1(\varphi_{NS})<0$. So, according to the theorem, the Neimark-Sacker bifurcation is super-critical.

5. Conclusion

A discrete-time nonlinear financial model is investigated which is obtained with The Nonstandart Finite Differance Scheme. Explicit Flip and Neimark-Sacker criterion is used for the existence of Neimark-Sacker bifurcation at fixed point E_0. Also, the direction of bifurcation is determined via the applications of center monifold theory and bifurcation theory. After all of the theoretical analyze is approved by a numerical example. Phase portrait of 3D system is presented by Figure1 for $a = 1.8, b = 0.5, c = 0.35$, $\varphi_{NS} = 0{,}405$ with initial conditions $(x_0, y_0, z_0) = (0.4, 0.6, 0.8)$. Time series solutions for $a = 1.8, b = 0.5, c = 0.35$ is shown in Figure2. Scatter plot of xyz for $a = 1.8, b = 0.5, c = 0.35$ is shown by Figure3.

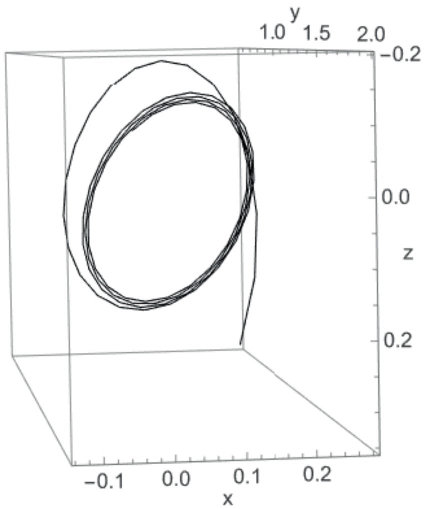
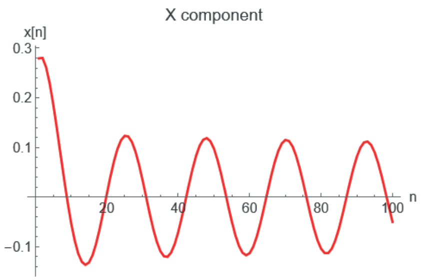
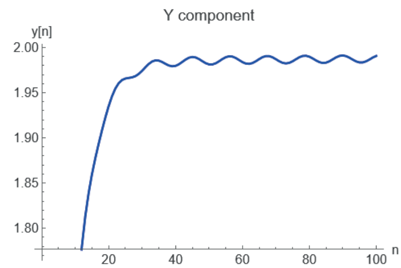


Figure1: Phase portrait for $a = 1.8, b = 0.5, c = 0.35$, $\varphi_{NS} = 0{,}405$ with initial conditions $(x_0, y_0, z_0) = (0.4, 0.6, 0.8)$.



(a)



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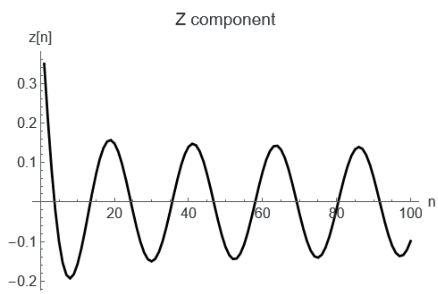

(c)

Figure 2: Time series solutions for $a = 1.8, b = 0.5, c = 0.35$, $\varphi_{NS} = 0{,}405$ with initial conditions $(x_0, y_0, z_0) = (0.4, 0.6, 0.8)$.

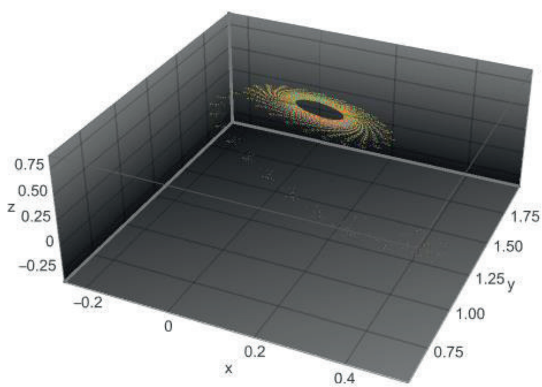

Figure 3: Scatter plot of xyz for $a = 1.8, b = 0.5, c = 0.35$, $\varphi_{NS} = 0{,}405$ with initial conditions $(x_0, y_0, z_0) = (0.4, 0.6, 0.8)$.

6. References

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