Chapter 2

Flow and Risk Allocation with Cooperative Game Theory in Flow Situations under Grey Uncertainty with Some Equal Surplus Sharing Approaches

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Abstract

In this study, multi-owner maximum flow problems defined under grey uncertainty are investigated within the framework of cooperative game theory. In this context, egalitarian distribution approaches are comprehensively addressed by considering the flow and risk allocation dimensions in flow problems. Grey interval numbers are used to model situations where arc capacities cannot be known precisely in real-life logistics and infrastructure networks. In this context, we aim to overcome the limitations of both deterministic and probabilistic approaches.

Based on a single-source, single-sink and multi-owner network structure, where arc capacities are expressed by grey interval numbers, two separate games are formulated for all possible coalitions: the grey maximum flow game $\langle N, v' \rangle$ and the grey risk game $\langle N, c' \rangle$. The grey characteristic function of each coalition is obtained using the BWC algorithm and the Grey REILP method. Through this method, a quantitative balance is established between risk and system return.

Four Equal Surplus Sharing Approaches, namely the Grey Banzhaf value, Grey CIS-value ($\mathcal{G}$CIS), Grey ENSC-value ($\mathcal{G}$ENSC), and Grey ED-value ($\mathcal{G}$ED) are applied to both games. The results are evaluated comparatively in terms of efficiency, individual rationality, and coalitional rationality criteria. The findings demonstrate that the $\mathcal{G}$ENSC-value lies within the grey core and ensures coalition stability. The results provide significant contributions to cooperative models for the design of fair, stable, and risk-sensitive collaboration in infrastructure and logistics networks under grey uncertainty.

Keywords: Grey numbers, Cooperative game theory, Maximum flow problem, Egalitarian distribution approach, Risk allocation, Grey uncertainty.

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https://doi.org/10.58830/ozgur.pub1361.c5492

1. Introduction

The most fundamental decision problem in transportation and logistics networks is maximizing the flow transmitted from a source node to a sink node. The maximum flow problem (MFP)is first solved by Ford and Fulkerson (1956) is used as a fundamental modeling tool in many engineering and management problems especially in electrical power systems, communication networks, computer networks and logistics networks (Asano & Asano, 2000). Although numerous effective solution algorithms have been developed for the deterministic MFP in the literature, the capacities of network components are often not known precisely in real-life applications. This uncertainty poses a critical decision difficulty, especially in structures where multiple operators control different network segments (Dantzig, 1955).

Various approaches have been proposed in the literature to model uncertainty, such as probability theory (Evans, 1976; Fishman, 1987; Goldberg & Tarjan, 1988), fuzzy set theory (Chanas & Kołodziejczyk, 1982; Chanas & Kołodziejczyk, 1984; Chanas & Kolodziejczyk, 1986; Ji et al., 2006), stochastic optimization (Hafezalkotob & Makui, 2015) and robust optimization (Ben-Tal et al., 2004; Minoux, 2009; Minoux, 2010). However, these methods have significant limitations. Estimating probability distributions relies on expert opinion and can lead to misleading results when evaluating rare events (Tversky & Kahneman, 1992). Fuzzy approaches require the selection of appropriate membership functions, whereas stochastic programming incurs high computational costs and may be infeasible (Hafezalkotob & Makui, 2015; Charnes & Cooper, 1959). In contrast, Interval Linear Programming (ILP) requires only the lower and upper bounds of the parameters, thus operating with less information and a lower computational burden, and with these features, it is becoming a preferred method in practical applications (Huang et al., 2013; Huang et al., 2015; Shaocheng, 1994).

In real life, different parts of the network may be owned by different individuals, companies, or countries. Natural gas pipelines, electrical transmission grids, and multi-operator logistics networks are typical examples of this situation (Koch et al., 2015; Tran et al., 2018). Collaboration among owners in such structures can significantly improve the network's overall performance. The fair allocation of the benefits from collaboration among the owners is the fundamental problem in cooperative game theory. Kalai and Zemel (1982a, 1982b) establish the theoretical foundation of flow games by revealing the structure of totally balanced games in multi-owner networks. Subsequently, Reyes (2005) perform fair flow allocation in transshipment networks via the Shapley value and Hafezalkotob and Makui (2015) solved the cooperative MFP under stochastic uncertainty, comparing different allocation concepts such as the Shapley value, the τ-value, and the core. Meanwhile, Baykasoğlu and Kubur Özbel (2019) solve the multi-owner maximum flow problem under interval uncertainty using the Risk-Explicit Interval Linear Programming (REILP) method and allocated both flow and risk among owners via the Shapley value.

On the other hand, grey system theory is introduced by Deng (1982) and has developed into a powerful tool for modeling systems that are partially known, have small sample sizes, or contain missing data (Deng, 1982). Grey numbers represent quantities whose exact values are unknown but whose containing intervals are known, and they naturally fit the structure of real-world data. Within this framework, Alparslan Gök et al. (2008, 2009, 2011) develope the fundamental concepts of cooperative grey game theory, while Olgun et al. (2016, 2017) define grey inventory games and examined cost allocation solutions. Dönmez et al. (2024) define egalitarian allocation solutions within the framework of grey numbers in grey inventory games and tested them on an application involving three arms factories.

The main contributions of this study can be summarized under three headings. First, the egalitarian allocation solution concepts used by Dönmez et al. (2024) in grey inventory games are applied to cooperative grey maximum flow games. Second, it is demonstrated that collaboration is a mechanism that not only increases the total flow but also reorganizes the uncertainty burden; two separate cooperative grey TU games, namely the grey flow game and the grey risk game are formulated, and simultaneous allocation is performed for both under four different fairness criteria. Third, it is numerically verified that the $\mathcal{G}$ENSC-value lies within the grey core, thereby demonstrating the transferability of the known compatibility between the core and ENSC (van den Brink & Funaki, 2009; Driessen & Funaki, 1991) for deterministic games to grey games.

The remainder of the chapter is organized as follows. In the second section, preliminary information regarding grey calculus and cooperative grey game theory is provided, while the grey egalitarian distribution framework is established. In the third section, a numerical application is presented for the logistics network example from Baykasoğlu and Kubur Özbel (2019). In the fourth section, the findings are discussed, theoretical and practical implications are presented, and future research directions are suggested.

2. Preliminaries

In this section, fundamental concepts related to grey system theory, cooperative grey game theory, and Equal Surplus Sharing Approaches used throughout the study are presented (Alparslan Gök et al., 2008; Dönmez et al., 2024; van den Brink & Funaki, 2009; Branzei et al., 2008).

2.1. Grey Arithmetic Operations

In this part, some grey calculus operations used in formulating cooperative grey games used in this chapter are presented. Numbers whose exact values are unknown but whose containing intervals are known are called grey numbers. For $\underline{a}, \overline{a} \in \mathbb{R}$, the number with lower bound $\underline{a}$ and upper bound $\overline{a}$ is denoted as $\otimes_a \in [\underline{a}, \overline{a}]$. For example, if the temperature of a room varies between 20 and 25 degrees, this situation is denoted as durum $\otimes_1 \in [20,25]$.

The set of all grey interval numbers in $\mathbb{R}$ is denoted by $\mathcal{G}(\mathbb{R})$, and for $\otimes_1, \otimes_2 \in \mathcal{G}(\mathbb{R})$, they are expressed as $\otimes_1 \in [\underline{a}, \overline{a}]$, $\otimes_2 \in [\underline{b}, \overline{b}]$, $|\otimes_1| = \underline{a} - \overline{a}$ and $\alpha \in \mathbb{R}_+$. Based on this, the following operations are defined:

1. $\otimes_1 + \otimes_2 \in [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$ (2.1)
2. $\alpha \otimes_1 \in [\alpha\underline{a}, \alpha\overline{a}]$. (2.2)

So, $\mathcal{G}(\mathbb{R})$ has cone structure.

In this study we use a partial subtraction operator. Our partial subtraction operator $\otimes_1 - \otimes_2$ is only defined when $|\overline{a} - \underline{a}| \geq |\overline{b} - \underline{b}|$ and is calculated as $\otimes_1 - \otimes_2 \in [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ (Alparslan Gök et al., 2009).

The ordered pair approach is adopted in ordering grey interval numbers. For $\otimes_1 \in [\underline{a}, \overline{a}]$ and $\otimes_2 \in [\underline{b}, \overline{b}]$, $[\underline{a}, \overline{a}]$ being better than $[\underline{b}, \overline{b}]$ even in a weak sense implies the satisfaction of the conditions $[\underline{a}, \overline{a}] \geq [\underline{b}, \overline{b}] \Leftrightarrow \underline{a} \geq \underline{b}$ ile $\overline{a} \geq \overline{b}$. The reverse case is denoted by $[\underline{a}, \overline{a}] \leq [\underline{b}, \overline{b}]$ and is shown as $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$ (Olgun et al., 2016).

In this context, fundamental concepts pertaining to cooperative grey game theory are now presented. A cooperative grey game is defined by the ordered pair $\langle N, w' \rangle$. Here, $N = \{1,2, \ldots, n\}$ is the set of players and $w': 2^N \to \mathcal{G}(\mathbb{R})$ denotes the characteristic function. Here, with $w'(\emptyset) \in [0,0]$, the value of the expected grey loss (cost) of coalition $S \in 2^N$, which is $w'(S) \in [\underline{w'(S)}, \overline{w'(S)}]$, is the grey loss function. While $\underline{w'(S)} \leq \overline{w'(S)}$, $\underline{w'(S)}$ and $\overline{w'(S)}$ indicate the minimum and maximum losses of the coalition. Therefore, a cooperative grey game can be considered as a classical cooperative game with $\otimes$ grey loss values. Grey solution methods are appropriate methods for solving loss (cost) problems using grey numbers for cooperative grey games. Grey solution methods and their elements are grey loss values defined in $G(\mathbb{R})$. $G(\mathbb{R})^N$ is defined as the set of all grey loss values, and $\mathcal{G}G^N$ is defined as the set of cooperative grey games.

Specifically, IG^N denotes the family of all cooperative interval games with player set N. For each $S \in 2^N$, when $w, w_1, w_2 \in IG^N$ and $w', w'_1, w'_2 \in GG^N$, if the condition $w'_1(S) \leq w'_2(S)$ is satisfied, we say $w'_1 \in w_1 \leq w'_2 \in w_2$. Here, $w'_1(S) \in w_1(S)$ and $w'_2(S) \in w_2(S)$ hold for each $S \in 2^N$. For $w'_1, w'_2 \in GG^N$ and $\lambda \in \mathbb{R}_+$, the games $(w'_1 + w'_2)$ and $(N, \lambda w')$ are defined by the relations $(w'_1 + w'_2)(S) = w'_1(S) + w'_2(S)$ and $(\lambda w')(S) = \lambda w'(S)$, respectively. Therefore GG^N is equipped with the $\leq$ ordering and has a cone structure with respect to addition and multiplication by non-negative scalars, as explained above. Furthermore, for $w'_1, w'_2 \in GG^N$, where $w'_1 \in w_1$ ve $w'_2 \in w_2$, when the condition $|w_1(S)| \geq |w_2(S)|$ is satisfied for each $S \in 2^N$, the game $(N, w'_1 - w'_2)$ is definied $(w'_1 - w'_2)(S) = w'_1(S) - w'_2(S) \in w_1(S) - w_2(S)$ (Alparslan Gök et al., 2019).

2.2. Equal Surplus Sharing Approaches

Four grey equal surplus sharing solution concepts for cooperative grey games are defined for Equal Surplus Sharing Approaches by utilizing the studies of van den Brink and Funaki (2009) and Dönmez et al. (2024).

The Grey Banzhaf value, grey *CIS*-value, grey *ENSC*-value, and grey *ED*-value solution are defined within the class $(SM\mathcal{G}G)^N$ just like the Grey Shapley value. The reason for this is that grey marginal operators are defined within the class $(SM\mathcal{G}G)^N$. The Grey Shapley value is defined and denoted as $\phi' : (SM\mathcal{G}G)^N \to G(\mathbb{R})^N$.

$$\phi_i'(c') = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} \sum_{i \in N} m_i^\sigma (c') = \frac{1}{n!} n!\ c'(N) = c'(N) \in \left[\underline{a_N}, \overline{a_N}\right] \qquad (2.3)$$

Definition 2.1. (Grey Banzhaf Value): The Banzhaf value assumes that each player joins any coalition with equal probability. This value is defined by $\beta' : (SM\mathcal{G}G)^N \to G(\mathbb{R}^N)$ and is expressed as

$$\beta_i'(c') \in \left[\underline{\beta_i'(c')}, \overline{\beta_i'(c')}\right] = \frac{1}{2^{N-1}} \sum_{i \in S} c'(S) - c'(S \setminus \{i\}), c' \in \left[\underline{a_N}, \overline{a_N}\right] \qquad (2.4)$$

This expression is valid for all $i \in N$ and all $c' \in (SM\mathcal{G}G)^N$.

Definition 2.2. ($\mathcal{G}$*CIS*-Value): The $\mathcal{G}CIS$-value allocates the individual grey value to each player and equally distributes the remainder of the grand coalition N among all players. The $\mathcal{G}CIS$-value is defined by $\mathcal{G}CIS' : (SM\mathcal{G}G)^N \to G(\mathbb{R}^N)$ and, where $|\ c'(N)\ | \leq \sum_{i \in N} c'(i)$, it is expressed as

$$\mathcal{G}CIS_i'(c') \in \left[\underline{\mathcal{G}CIS_i'(c')}, \overline{\mathcal{G}CIS_i'(c')}\right]$$
$$= c'(\{i\}) + \frac{1}{|\ N\ |} \left(c'(N) - \sum_{j \in N} c'(\{j\}) \right), c' \in \left[\underline{a_N}, \overline{a_N}\right] \qquad (2.5)$$

This expression is valid for all $i \in N$ and all $c' \in (SM\mathcal{G}G)^N$.

Definition 2.3. ($\mathcal{G}$*ENSC*-Value): In a grey game extending from c' to $c'^* \in SM\mathcal{G}G^N$ in the class $SM\mathcal{G}G^N$, each coalition $S \subseteq N$ is assigned the grey value that the grand coalition N lose if coalition S are to leave N. For each $S \subseteq N$, it is defined as $c^{*\prime}(S) = c'(S) - c'(N \setminus S)$. The Grey *ENSC*-value ($\mathcal{G}ENSC$-value), assigns the *CIS*-value of the dual game c'^* to each game c'. It is defined by $\mathcal{G}ENSC' : SM\mathcal{G}G^N \to G(\mathbb{R}^N)$ and expressed as

$$\mathcal{G}ENSC_i'(c') \in \left[\underline{\mathcal{G}ENSC_i'(c')}, \overline{\mathcal{G}ENSC_i'(c')}\right] = \mathcal{G}CIS_i'(c'^*) \in \left[\underline{\mathcal{G}CIS_i'(c'^*)}, \overline{\mathcal{G}CIS_i'(c'^*)}\right]$$
$$= \frac{1}{|\ N\ |} \left(c'(N) + \sum_{j \in N} c'(N \setminus \{j\}) \right) - c'(N), c' \in \left[\underline{a_N}, \overline{a_N}\right] \qquad (2.6)$$

This expression is valid for all $i \in N$ and all $c' \in SM\mathcal{G}G^N$. It is observed that $\mid c'(N) + \sum_{j\in N} c'\,(N \setminus \{j\}) \mid \leq \mid N \mid\mid c'(N \setminus \{i\})$. Therefore, the *GENSC*-value determines the grey marginal contribution of each player to the grand coalition and distributes the remainder equally among the players.

Definition 2.4. ($\mathcal{G}ED$-Value): The Grey *ED*-value ($\mathcal{G}ED$-value) is defined by $\mathcal{G}ED' : \mathcal{G}G^N \to G(\mathbb{R}^N)$ and is expressed as

$$\mathcal{G}ED_i'(c') \in \left[\underline{\mathcal{G}ED_i'(c')}, \overline{\mathcal{G}ED_i'(c')}\right] = \frac{c'(N)}{\mid N \mid}, c' \in \left[\underline{a_N}, \overline{a_N}\right] \tag{2.7}$$

This expression is valid for all $i \in N$ and all $c' \in \mathcal{G}G^N$.

3. Formulation of Maximum Flow Problems under Grey Uncertainty

In this section, the multi-owner maximum flow problem under grey uncertainty is expressed within the framework of cooperative grey game theory. The model is built on a three-layered structure: ***i.*** definition of the network structure with grey arc capacities, ***ii.*** formulation of the grey characteristic function for each coalition via the BWC algorithm and Grey REILP method, and ***iii.*** transformation of the obtained grey characteristic functions into the grey flow game $\langle N, v' \rangle$ and the grey risk game $\langle N, c' \rangle$.

3.1. Network Structure and Grey Capacity Model

Consider an acyclic directed graph $G(A, V)$ with a single source node s and a single sink node t. Here, A denotes the finite set of arcs, and V denotes the finite set of nodes. Let $N = \{1,2, \dots, n\}$ be the set of players; the set of arcs under the control of the k-th player is denoted by $A_{\{k\}}$, and they are united as $A = \bigcup_{k=1}^{n} A_{\{k\}}$. It is assumed that the arc sets of the players are disjoint, meaning $A_{\{k\}} \cap A_{\{l\}} = \emptyset$ when $k \neq l$.

In real-life applications, arc capacities are affected by structural sources of uncertainty such as weather conditions, equipment breakdowns, demand fluctuations and measurement errors. These capacities cannot be known precisely.

Definition 3.1. (Network Under Grey Uncertainty): When the capacity of each arc $(i, j) \in A$ is expressed by a grey interval number, we have

$$\otimes_{cap_{ij}} \in \left[\underline{cap_{ij}}, \overline{cap_{ij}}\right] \in G(\mathbb{R}),\ \underline{cap_{ij}} \leq \overline{cap_{ij}}. \tag{3.1}$$

(3.1) is called a network under grey uncertainty. Here, $\underline{cap_{ij}}$ represents the most pessimistic realization value of the capacity, and $\overline{cap_{ij}}$ represents the most optimistic one.

3.2. Coalition-Based Grey Maximum Flow Model

For any coalition $v_m \subseteq N$, the set of arcs used by the coalition is defined as $A_{v_m} = \bigcup_{k \in v_m} A_{\{k\}}$. The maximum flow problem for this coalition is formulated with the following grey ILP ($\mathcal{G}$ILP) model.

The objective function is defined by

$$\max \text{ flow } (v_m) = \sum_{j \in F_s} \otimes_{x_{sj}} = \sum_{i \in E_t} \otimes_{x_{it}}. \qquad (3.2)$$

Here, the capacity constraint for the flow on each arc $\forall(i,j) \in A_{v_m}$ cannot exceed the grey capacity of that arc with $\otimes_{x_{ij}} \preceq \otimes_{cap_{ij}}$. In the flow conservation constraint, for all intermediate nodes other than source s and sink t, when $\forall i \in V \setminus \{s,t\}$, we have

$$\sum_{l \in E_i} \otimes_{x_{li}} = \sum_{l \in F_i} \otimes_{x_{il}} \, . \qquad (3.3)$$

The amount of incoming flow must equal the amount of outgoing flow. The non-negativity constraint is expressed as $\otimes_{x_{ij}} \succeq [0,0]$ when $\forall(i,j) \in A_{v_m}$, which establishes that the model is a $\mathcal{G}$ILP.

3.3. Solution of $\mathcal{G}$ILP with BWC Algorithm and Obtaining the Grey Characteristic Function

The $\mathcal{G}$ILP model is solved with the Best-Worst Case (BWC) algorithm proposed by Shaocheng (1994). This algorithm decomposes the grey model into two deterministic sub-models that represent the two extreme uncertainty conditions.

Most Optimistic Case (Upper Bound) is given by

$$\max f^{+} = \sum_{(i,j) \in A_{v_m}} \overline{cap_{ij}} \cdot z_{ij} \qquad (3.4)$$

$$\sum_{j \in F_s} x_{sj}^{+} = \sum_{i \in E_t} x_{it}^{+} \qquad (3.5)$$

$$x_{ij}^{+} \leq \overline{cap_{ij}}, \sum_{l \in E_i} x_{li}^{+} = \sum_{l \in F_i} x_{il}^{+}, x_{ij}^{+} \geq 0. \qquad (3.6)$$

The most optimistic case implies that the constraints define the widest solution space.

Most Pessimistic Case (Lower Bound) is given by

$$\max f^{-} = \sum_{(i,j) \in A_{v_m}} \underline{cap_{ij}} \cdot z_{ij} \qquad (3.7)$$

$$\sum_{j\in F_s} x_{sj}^- = \sum_{i\in E_t} x_{it}^- \tag{3.8}$$

$$x_{ij}^- \leq \underline{cap}_{ij}, \sum_{l\in E_i} x_{li}^- = \sum_{l\in F_i} x_{il}^-, x_{ij}^- \geq 0. \tag{3.9}$$

This most pessimistic case implies that the constraints define the narrowest solution space. From the solution of the two sub-models, the grey optimal flow value is obtained for each coalition v_m.

$$f_{opt}^{\pm}(v_m) \in \left[f_{opt}^{-}(v_m), f_{opt}^{+}(v_m)\right] \in G(\mathbb{R}) \tag{3.10}$$

The solution corresponding to f_{opt}^{-} indicates zero risk of constraint violation since it is obtained under the most pessimistic condition, where constraints are defined in the narrowest manner. The solution corresponding to f_{opt}^{+}, carries a high risk of violation in the event that uncertainty is realized, despite the capacities taking their widest values. The actual system will operate somewhere between these two extremes. This feature demonstrates that the grey interval is both a boundary of information and a carrier of risk.

3.4. Formulation of the $\mathcal{G}$REILP Model

The classical $\mathcal{G}$ILP approach has two main limitations: first, the obtained interval solutions can lead to infeasible or non-optimal decisions in practice and second, a quantitative link cannot be established between decision risk and system payoff (Zou et al., 2010). In order to overcome these limitations, the REILP approach (Zou et al., 2010) is adapted to the cooperative grey game framework.

Definition 3.2. (Grey Risk Function): For each arc (i,j) in A_{c_m}, the grey risk function is defined as follows, where $r_k \in [0,1]$ and $\forall(i,j) \in A_{c_m}$.

$$\xi_k = r_k \cdot \frac{\overline{cap_{ij}} - \underline{cap_{ij}}}{\overline{cap_{ij}} + \underline{cap_{ij}}} \tag{3.11}$$

Here, $r_k \in [0,1]$ represents the risk variable belonging to the k-th arc, the numerator $\left(\overline{cap_{ij}} - \underline{cap_{ij}}\right)$ represents the grey uncertainty width of the arc, and the denominator $\left(\overline{cap_{ij}} + \underline{cap_{ij}}\right)$ represents the center value. This normalization brings the risk contributions of arcs with capacities of different magnitudes to a comparable scale. When $\xi_k = 0$, $r_k = 0$, meaning the capacity limit for the given arc is defined by its most pessimistic value, and there is no risk of constraint violation. When $\xi_k > 0$, $r_k > 0$, the probability of constraint violation increases as the upper limit of the capacity is approached.

Definition 3.3. ($\mathcal{G}$REILP Model): For each coalition $c_m \subseteq N$, the $\mathcal{G}$REILP model is defined as follows:

$$\min \xi(c_m) = \sum_{(i,j)\in A_{c_m}} r_k \cdot \frac{\overline{cap_{ij}} - \underline{cap_{ij}}}{\overline{cap_{ij}} + \underline{cap_{ij}}} \tag{3.12}$$

$$\sum_{j\in F_s} x_{sj} \geq f_{opt}^{-}(c_m) + \lambda_0 \left(f_{opt}^{+}(c_m) - f_{opt}^{-}(c_m)\right) \tag{3.13}$$

$$x_{ij} - \underline{cap_{ij}} \leq \left(\overline{cap_{ij}} - \underline{cap_{ij}}\right) \cdot r_k, \forall (i,j) \in A_{c_m} \tag{3.14}$$

$$\sum_{l\in E_i} x_{li} = \sum_{l\in F_i} x_{il}, \forall i \in V \setminus \{s,t\} \tag{3.15}$$

$\lambda_0 = \lambda_{pre}$, $\forall k$ iken $0 \leq r_k \leq 1$ ve $x_{ij} \geq 0$.

Definition 3.4. (Aspiration Level): $\lambda_0 \in [0,1]$ is the aspiration level that reflects the decision maker's attitude towards risk.

$\lambda_0 = 0$: The most pessimistic case; the system payoff is equal to f_{opt}^{-} and the risk is zero.

$\lambda_0 = 1$: The most optimistic case; the system payoff reaches f_{opt}^{+} and the risk is at a maximum level.

$0 < \lambda_0 < 1$: The decision maker strikes a balance between risk and payoff.

Definition 3.5. (Normalized Risk Level): In order to compare the risk values of different coalitions and different aspiration levels, the normalized risk level is defined.

$$NRL(c_m, \lambda_0) = \frac{\xi(c_m, \lambda_0) - \xi_{min}(c_m)}{\xi_{min}(c_m) - \xi_{min}(c_m)} \tag{3.16}$$

Here, $\xi_{min}(c_m) = \xi(c_m, \lambda_0 = 0)$ and $\xi_{max}(c_m) = \xi(c_m, \lambda_0 = 1)$.

$NRL = 0$ represents the most pessimistic case (zero risk), while $NRL = 1$ represents the most optimistic case (maximum risk).

3.5. Formulation of Cooperative Grey Games

The $\mathcal{G}$REILP model is solved at the values of $\lambda_0 = 0$ and $\lambda_0 = 1$ for each coalition $c_m \subseteq N$, directly obtaining the lower and upper bounds of the grey characteristic functions. This step is an important structure connecting REILP solutions to cooperative grey game theory.

Definition 3.6. (Grey Maximum Flow Game): The grey maximum flow game $\langle N, v' \rangle$ is the game $\langle N, v' \rangle \in \mathcal{IG}^N$ where, with $N = \{1,2,\ldots,n\}$, the grey characteristic function for each $c_m \subseteq N$ is defined as $v'(c_m) \in \left[f_{opt}^{-}(c_m), f_{opt}^{+}(c_m)\right] \in G(\mathbb{R})$. It is clear that $v'(\emptyset) = [0,0]$.

Definition 3.7. (Grey Risk Game): The grey risk game $\langle N, c' \rangle$ is the game $\langle N, c' \rangle \in \mathcal{IG}^N$ where the grey characteristic function for each $c_m \subseteq N$ is defined as

$$c'(c_m) \in [\xi(c_m, \lambda_0 = 0), \xi c_m, \lambda_0 = 1)] = [\xi_{min}(c_m), \xi_{max}(c_m)] \in G(\mathbb{R}). \quad (3.17)$$

4. Application

In this section, the multi-owner logistics network problem addressed by Baykasoğlu and Kubur Özbel (2019) is adopted as a numerical example. In the aforementioned study, a directed network structure with a single source and a single sink node, controlled by three different owners where $N = \{1,2,3\}$ is examined. As seen in Figure 1, the arc capacities in the network contain uncertainty and are represented by interval numbers.

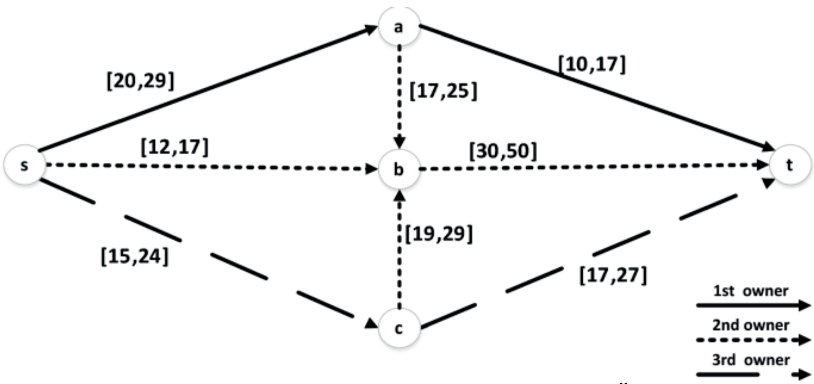

Figure 1. Logistics Network (Baykasoğlu & Kubur Özbel, 2019)

Maximum flow values for all possible coalitions are reported by Baykasoğlu and Kubur Özbel (2019). These values are taken directly to form the lower and upper bounds of the grey characteristic function and are summarized in Table 1.

Table 1. Grey maximum flow and risk values of all coalitions taken from Baykasoğlu and Kubur Özbel (2019)

$\{S\}$	$v'(\{S\})$	$c'(\{S\})$
$\{1\}$	$v'(\{1\}) \in [10,17]$	$c'(\{1\}) \in [0,0.25]$
$\{2\}$	$v'(\{2\}) \in [12,17]$	$c'(\{2\}) \in [0,0.17]$
$\{3\}$	$v'(\{3\}) \in [15,24]$	$c'(\{3\}) \in [0,0.38]$
$\{1,2\}$	$v'(\{1,2\}) \in [32,46]$	$c'(\{1,2\}) \in [0,0.47]$
$\{1,3\}$	$v'(\{1,3\}) \in [25,41]$	$c'(\{1,3\}) \in [0,0.64]$
$\{2,3\}$	$v'(\{2,3\}) \in [27,41]$	$c'(\{2,3\}) \in [0,0.40]$
$\{1,2,3\}$	$v'(\{1,2,3\}) \in [47,70]$	$c'(\{1,2,3\}) \in [0,0.79]$

4.1. Grey Banzhaf Value

For the Grey Banzhaf, the equation defined in (2.4) will be used in both flow and risk calculations.

Grey Banzhaf Value for Flow:

$$\beta_1'(v') \in \left[\underline{\beta_1'(v')}, \overline{\beta_1'(v')}\right] = \frac{1}{4}[(v'(\{1\}) - v'(\emptyset)) + (v'(\{1,2\}) - v'(\{2\})) + (v'(\{1,3\}) - v'(\{3\})) + (v'(N) - v'(\{2,3\}))]$$

$$= \frac{1}{4}[[10,17] + ([32,46] - [12,17]) + ([25,41] - [15,24]) + ([47,70] - [27,41])]$$

$$= \frac{1}{4}[[10,17] + [20,29] + [10,17] + [20,29]]$$

$$= \frac{1}{4}[60,92] = [15,23]$$

$$\beta_2'(v') \in \left[\underline{\beta_2'(v')}, \overline{\beta_2'(v')}\right] = \frac{1}{4}[v'(\{2\}) + (v'(\{1,2\}) - v'(\{1\})) + (v'(\{2,3\}) - v'(\{3\})) + (v'(N) - v'(\{1,3\}))]$$

$$= \frac{1}{4}[[12,17] + [20,29] + [12,17] + [22,29]]$$

$$= \frac{1}{4}[66,92] = [16.5,23]$$

$$\beta_3'(v') \in \left[\underline{\beta_3'(v')}, \overline{\beta_3'(v')}\right]$$
$$= \frac{1}{4}[v'(\{3\}) + (v'(\{1,3\}) - v'(\{1\})) + (v'(\{2,3\}) - v'(\{2\})) + (v'(N) - v'(\{1,2\}))]$$
$$= \frac{1}{4}[[15,24] + [15,24] + [15,24] + [15,24]]$$
$$= \frac{1}{4}[60,96] = [15,24]$$
$$\beta_i'(v') \in ([15,23], [16.5,23], [15,24])$$

Grey Banzhaf Value for Risk:

$$\beta_1'(c') \in \left[\underline{\beta_1'(c')}, \overline{\beta_1'(c')}\right]$$
$$= \frac{1}{4}[[0,0.25] + ([0,0.47] - [0,0.17]) + ([0,0.64] - [0,0.38]) + ([0,0.79] - [0,0.40])]$$
$$= \frac{1}{4}[[0,0.25] + [0,0.30] + [0,0.26] + [0,0.39]]$$
$$= \frac{1}{4}[0,1.20] = [0,0.3]$$

$$\beta_2'(c') \in \left[\underline{\beta_2'(c')}, \overline{\beta_2'(c')}\right]$$
$$= \frac{1}{4}[[0,0.17] + ([0,0.47] - [0,0.25]) + ([0,0.40] - [0,0.38]) + ([0,0.79] - [0,0.64])]$$
$$= \frac{1}{4}[[0,0.17] + [0,0.22] + [0,0.02] + [0,0.15]]$$
$$= \frac{1}{4}[0,0.56] = [0,0.14]$$

$$\beta_3'(c') \in \left[\underline{\beta_3'(c')}, \overline{\beta_3'(c')}\right]$$
$$= \frac{1}{4}[[0,0.38] + ([0,0.64] - [0,0.25]) + ([0,0.40] - [0,0.17]) + ([0,0.79] - [0,0.47])]$$
$$= \frac{1}{4}[[0,0.38] + [0,0.39] + [0,0.23] + [0,0.32]]$$
$$= \frac{1}{4}[0,1.32] = [0,0.33]$$

4.2. Grey CIS-Value

For the Grey CIS-Value, the equation defined in (2.5) will be used in both flow and risk calculations.

Grey CIS-Value for Flow:

$$\mathcal{G}CIS'_1(v') \in \left[\underline{\mathcal{G}CIS'_1(v')}, \overline{\mathcal{G}CIS'_1(v')}\right]$$
$$= [10,17] + \frac{1}{3}\big([47,70] - ([10,17] + [12,17] + [15,24])\big)$$
$$= [10,17] + [3.33,4] = [13.33,21]$$

$$\mathcal{G}CIS'_2(v') \in \left[\underline{\mathcal{G}CIS'_2(v')}, \overline{\mathcal{G}CIS'_2(v')}\right]$$
$$= [12,17] + \frac{1}{3}\big([47,70] - ([10,17] + [12,17] + [15,24])\big)$$
$$= [10,17] + [3.33,4] = [15.33,21]$$

$$\mathcal{G}CIS'_3(v') \in \left[\underline{\mathcal{G}CIS'_3(v')}, \overline{\mathcal{G}CIS'_3(v')}\right]$$
$$= [15,24] + \frac{1}{3}\big([47,70] - ([10,17] + [12,17] + [15,24])\big)$$
$$= [15,24] + [3.33,4] = [18.33,28]$$

Grey CIS-Value for Risk:

$$\mathcal{G}CIS'_1(c') \in \left[\underline{\mathcal{G}CIS'_1(c')}, \overline{\mathcal{G}CIS'_1(c')}\right]$$
$$= [0,0.25] + \frac{1}{3}\big([0,0.79] - ([0,0.25] + [0,0.17] + [0,0.38])\big)$$
$$= [0,0.25] + [0,-0.003] = [0,0.247]$$

$$\mathcal{G}CIS'_2(c') \in \left[\underline{\mathcal{G}CIS'_2(c')}, \overline{\mathcal{G}CIS'_2(c')}\right]$$
$$= [0,0.17] + \frac{1}{3}\big([0,0.79] - ([0,0.25] + [0,0.17] + [0,0.38])\big)$$
$$= [0,0.17] + [0,-0.003] = [0,0.167]$$

$$\mathcal{G}CIS'_3(c') \in \left[\underline{\mathcal{G}CIS'_3(c')}, \overline{\mathcal{G}CIS'_3(c')}\right]$$
$$= [0,0.38] + \frac{1}{3}\big([0,0.79] - ([0,0.25] + [0,0.17] + [0,0.38])\big)$$
$$= [0,0.38] + [0,-0.003] = [0,0.377]$$

4.3. Grey ENSC-Value

For the Grey ENSC-Value, the equation defined in (2.6) will be used in both flow and risk calculations.

Grey ENSC-Value for Flow:

$$\mathcal{G}ENSC'_1(v') \in \left[\underline{\mathcal{G}ENSC'_1(v')}, \overline{\mathcal{G}ENSC'_1(v')}\right] = \frac{1}{3}\big([47,70] + ([32,46] + [25,41] + [27,41])\big) - [27,41] = [43.67,66] - [27,41] = [16.67,25]$$

$$\mathcal{G}ENSC'_2(v') \in \left[\underline{\mathcal{G}ENSC'_2(v')}, \overline{\mathcal{G}ENSC'_2(v')}\right] = \frac{1}{3}\big([47,70] + ([32,46] + [25,41] + [27,41])\big) - [25,41] = [43.67,66] - [25,41] = [18.67,25]$$

$$\mathcal{G}ENSC'_3(v') \in \left[\underline{\mathcal{G}ENSC'_3(v')}, \overline{\mathcal{G}ENSC'_3(v')}\right] = \frac{1}{3}\big([47,70] + ([32,46] + [25,41] + [27,41])\big) - [32,46] = [43.67,66] - [32,46] = [11.67,20]$$

Grey ENSC-Value for Risk:

$$\mathcal{G}ENSC'_1(c') \in \left[\underline{\mathcal{G}ENSC'_1(c')}, \overline{\mathcal{G}ENSC'_1(c')}\right] = \frac{1}{3}\big([0,0.79] + ([0,0.40] + [0,0.64] + [0,0.47])\big) - [0,0.40] = [0,0.767] - [0,0.40] = [0,0.367]$$

$$\mathcal{G}ENSC'_2(c') \in \left[\underline{\mathcal{G}ENSC'_2(c')}, \overline{\mathcal{G}ENSC'_2(c')}\right] = \frac{1}{3}\big([0,0.79] + ([0,0.40] + [0,0.64] + [0,0.47])\big) - [0,0.64] = [0,0.767] - [0,0.64] = [0,0.127]$$

$$\mathcal{G}ENSC'_3(c') \in \left[\underline{\mathcal{G}ENSC'_3(c')}, \overline{\mathcal{G}ENSC'_3(c')}\right] = \frac{1}{3}\big([0,0.79] + ([0,0.40] + [0,0.64] + [0,0.47])\big) - [0,0.47] = [0,0.767] - [0,0.47] = [0,0.297]$$

4.4. Grey ED-Value

For the Grey ED-Value, the equation defined in (2.7) will be used in both flow and risk calculations.

$$\mathcal{G}ED_i'(v') \in \left[\underline{\mathcal{G}ED_i'(v')}, \overline{\mathcal{G}ED_i'(v')}\right] = \frac{v'(N)}{|N|} = \frac{[47,70]}{3} = [15.67,23.33]$$

$$\mathcal{G}ED_i'(c') \in \left[\underline{\mathcal{G}ED_i'(c')}, \overline{\mathcal{G}ED_i'(c')}\right] = \frac{c'(N)}{|N|} = \frac{[0,0.79]}{3} = [0,0.263]$$

Following the calculation of grey Equal Surplus Sharing Approaches for flow and risk, all coalition values and results for the four Equal Surplus Sharing Solutions are obtained. A comprehensive evaluation of these results is presented below.

It has been demonstrated that the grey flow game $\langle N, v' \rangle$ is superadditive as follows:

$$v'(\{1,3\}) = [25,41] \geq [10,17] + [15,24] = [25,41]$$
$$v'(\{1,2,3\}) = [47,70] \geq [10,17] + [12,17] + [15,24] = [37,58].$$

This result mathematically demonstrates that the participation of each owner in the coalition yields strictly better results compared to acting alone. This feature guarantees that forming coalitions is a rational strategy in real-life applications such as logistics networks and natural gas transmission systems in the literature (Koch et al., 2015; Tran et al., 2018). Numerical results also support this finding. While the grey flow values of individual players are [10,17], [12,17] and [15,24] respectively, the value of the grand coalition increases to the level of [47,70]. This increase indicates that collaborative operations expand network utilization across complementary arcs and reduce inefficiencies stemming from fragmented ownership structures. From the perspective of application areas, this result indicates that total system output can increase significantly by either removing institutional boundaries or strengthening coordination protocols in transportation, energy transmission and supply networks are controlled by multiple operators.

The $\mathcal{G}CIS$, $\mathcal{G}ENSC$ and $\mathcal{G}ED$ values satisfy the efficiency condition:

$$\mathcal{G}CIS_1'(v') + \mathcal{G}CIS_2'(v') + \mathcal{G}CIS_3'(v') \in \left[\underline{\mathcal{G}CIS_1'(v')}, \overline{\mathcal{G}CIS_1'(v')}\right] + \left[\underline{\mathcal{G}CIS_2'(v')}, \overline{\mathcal{G}CIS_2'(v')}\right] + \left[\underline{\mathcal{G}CIS_3'(v')}, \overline{\mathcal{G}CIS_3'(v')}\right] = [47,70] = v'(\{1,2,3\}),$$

$$\mathcal{G}ENSC'_1(v') + \mathcal{G}ENSC'_2(v') + \mathcal{G}ENSC'_3(v') \in \left[\underline{\mathcal{G}ENSC'_1(v')}, \overline{\mathcal{G}ENSC'_1(v')}\right] + \left[\underline{\mathcal{G}ENSC'_2(v')}, \overline{\mathcal{G}ENSC'_2(v')}\right] + \left[\underline{\mathcal{G}ENSC'_3(v')}, \overline{\mathcal{G}ENSC'_3(v')}\right] = [47{,}70] = v'(\{1{,}2{,}3\}),$$

$$\mathcal{G}ED'_1(v') + \mathcal{G}ED'_2(v') + \mathcal{G}ED'_2(v') \in \left[\underline{\mathcal{G}ED'_1(v')}, \overline{\mathcal{G}ED'_1(v')}\right] + \left[\underline{\mathcal{G}ED'_2(v')}, \overline{\mathcal{G}ED'_2(v')}\right] + \left[\underline{\mathcal{G}ED'_3(v')}, \overline{\mathcal{G}ED'_3(v')}\right] = [47{,}70] = v'(\{1{,}2{,}3\}).$$

The Grey Banzhaf value is obtained in a way that does not satisfy the efficiency property

$$\beta'_1(v') + \beta'_2(v') + \beta'_3(v') \in \left[\underline{\beta'_1(v')}, \overline{\beta'_1(v')}\right] + \left[\underline{\beta'_2(v')}, \overline{\beta'_2(v')}\right] + \left[\underline{\beta'_3(v')}, \overline{\beta'_3(v')}\right] = [45.5{,}70] \neq [47{,}70] = v'(\{1{,}2{,}3\})$$

This demonstrates that the known property of the Banzhaf value in deterministic games carries over to grey games (Branzei et al., 2008) and confirms that this property is preserved within the grey framework as well.

The verification of the core condition reveals a significant divergence among the solution concepts. Although the $\mathcal{G}$CIS-value satisfies individual rationality:

$$\mathcal{G}CIS'_1(v') \in \left[\underline{\mathcal{G}CIS'_1(v')}, \overline{\mathcal{G}CIS'_1(v')}\right] = [13.33{,}21] \succeq v'(\{1\}) = [10{,}17],$$
$$\mathcal{G}CIS'_2(v') \in \left[\underline{\mathcal{G}CIS'_2(v')}, \overline{\mathcal{G}CIS'_2(v')}\right] = [15.33{,}21] \succeq v'(\{2\}) = [12{,}17],$$
$$\mathcal{G}CIS'_3(v') \in \left[\underline{\mathcal{G}CIS'_3(v')}, \overline{\mathcal{G}CIS'_3(v')}\right] = [18.33{,}28] \succeq v'(\{3\}) = [15{,}24].$$

It violates coalitional rationality:

$$\mathcal{G}CIS'_1(v') + \mathcal{G}CIS'_2(v') \in \left[\underline{\mathcal{G}CIS'_1(v')}, \overline{\mathcal{G}CIS'_1(v')}\right] + \left[\underline{\mathcal{G}CIS'_2(v')}, \overline{\mathcal{G}CIS'_2(v')}\right] = [28.66{,}42] \succeq v'(\{1{,}2\}) = [32{,}46].$$

The inequality $28.66 < 32$ at the lower bound indicates that the two-owner coalition may have an incentive to leave the grand coalition. In contrast, the $\mathcal{G}ENSC$-value satisfies all coalitional rationality conditions.

$$\mathcal{G}ENSC'_1(v') + \mathcal{G}ENSC'_2(v') \in \left[\underline{\mathcal{G}ENSC'_1(v')}, \overline{\mathcal{G}ENSC'_1(v')}\right] + \left[\underline{\mathcal{G}ENSC'_2(v')}, \overline{\mathcal{G}ENSC'_2(v')}\right] = [35.34{,}50] \succeq [32{,}46],$$

$$\mathcal{G}ENSC_1'(v') + \mathcal{G}ENSC_3'(v') \in \left[\underline{\mathcal{G}ENSC_1'(v')}, \overline{\mathcal{G}ENSC_1'(v')}\right] + \left[\underline{\mathcal{G}ENSC_3'(v')}, \overline{\mathcal{G}ENSC_3'(v')}\right] = [28.34,45] \succeq [25,41],$$

$$\mathcal{G}ENSC_2'(v') + \mathcal{G}ENSC_3'(v') \in \left[\underline{\mathcal{G}ENSC_2'(v')}, \overline{\mathcal{G}ENSC_2'(v')}\right] + \left[\underline{\mathcal{G}ENSC_3'(v')}, \overline{\mathcal{G}ENSC_3'(v')}\right] = [30.34,45] \succeq [27,41].$$

This finding mathematically demonstrates that the $\mathcal{G}ENSC$-value lies within the grey core and that no sub-coalition has a rational justification for leaving the grand coalition. This property, known in the literature as deterministic games (van den Brink & Funaki, 2009; Driessen & Funaki, 1991), has been demonstrated in the grey flow game in this study.

Based on this, the solutions of the egalitarian distribution methods calculated for flow and risk are presented in Table 2 and Table 3 and the whitened flow and risk values are presented in Table 4 and Table 5.

Table 2. Results of egalitarian distribution methods in the grey flow game

	$\boldsymbol{\phi_1}$	$\boldsymbol{\phi_2}$	$\boldsymbol{\phi_3}$	**Total**	**Core**
$\boldsymbol{\beta_i'(v')}$	$\in [15,23]$	$\in [16.5,23]$	$\in [15,24]$	$\in [46.5,70]$	**Partial**
$\boldsymbol{\mathcal{G}CIS_i'(v')}$	$\in [13.33,21]$	$\in [15.33,21]$	$\in [18.33,28]$	$\in [47,70]$	✕
$\boldsymbol{\mathcal{G}ENSC_i'(v')}$	$\in [16.67,25]$	$\in [18.67,25]$	$\in [11.67,20]$	$\in [47,70]$	✔
$\boldsymbol{\mathcal{G}ED_i'(v')}$	$\in [15.67,23.33]$	$\in [15.67,23.33]$	$\in [15.67,23.33]$	$\in [47,70]$	**Partial**

Table 3. Results of egalitarian distribution methods in the grey risk game

	$\boldsymbol{\phi_1}$	$\boldsymbol{\phi_2}$	$\boldsymbol{\phi_3}$	**Total**
$\boldsymbol{\beta_i'(c')}$	$\in [0,0.300]$	$\in [0,0.140]$	$\in [0,0.330]$	$\in [0,0.770]$
$\boldsymbol{\mathcal{G}CIS_i'(c')}$	$\in [0,0.247]$	$\in [0,0.167]$	$\in [0,0.377]$	$\in [0,0.791]$
$\boldsymbol{\mathcal{G}ENSC_i'(c')}$	$\in [0,0.367]$	$\in [0,0.127]$	$\in [0,0.297]$	$\in [0,0.791]$
$\boldsymbol{\mathcal{G}ED_i'(c')}$	$\in [0,0.263]$	$\in [0,0.263]$	$\in [0,0263]$	$\in [0,0.789]$

Crisp Comparison with Whitening ($\alpha = 0.5$)

To facilitate the interpretation of the results, whitening values with $\alpha = 1/2$ are calculated using the formula

$$\boldsymbol{\phi}_i' = \frac{\underline{\boldsymbol{\phi}_i'} + \overline{\boldsymbol{\phi}_i'}}{2} \tag{4.1}$$

are shown in Table 4 and Table 5.

Table 4. Whitened flow distribution results

	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}_2$	$\boldsymbol{\phi}_3$	**Total**
$\boldsymbol{\beta}_i'(\boldsymbol{v}')$	19.00	19.75	19.50	58.25
$\boldsymbol{\mathcal{G}CIS}_i'(\boldsymbol{v}')$	17.17	18.17	23.17	58.50
$\boldsymbol{\mathcal{G}ENSC}_i'(\boldsymbol{v}')$	20.83	21.83	15.83	58.50
$\boldsymbol{\mathcal{G}ED}_i'(\boldsymbol{v}')$	19.50	19.50	19.50	58.50

Table 5. Whitened risk distribution results

	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}_2$	$\boldsymbol{\phi}_3$	**Total**
$\boldsymbol{\beta}_i'(\boldsymbol{c}')$	0.150	0.070	0.165	0.385
$\boldsymbol{\mathcal{G}CIS}_i'(\boldsymbol{c}')$	0.124	0.084	0.189	0.396
$\boldsymbol{\mathcal{G}ENSC}_i'(\boldsymbol{c}')$	0.184	0.064	0.149	0.396
$\boldsymbol{\mathcal{G}ED}_i'(\boldsymbol{c}')$	0.132	0.132	0.132	0.395

5. An Application

In this study, Equal Surplus Sharing Approaches for cooperative maximum flow problems defined under grey uncertainty are addressed through flow and risk allocation. A framework has been established that allows the distribution of both network performance and the risk burden arising from uncertainty within the same cooperative structure. The fundamental contribution of the study is to demonstrate that collaboration is a mechanism that not only increases the total flow but also reorganizes the uncertainty burden by coupling collaboration, which is evaluated solely based on coalitional flow capacity in classical or uncertain interval maximum flow games, with risk magnitudes expressed by grey numbers.

The theoretical foundation of the model has been established with grey arithmetic operations, the partial subtraction operator, and the solution concepts of Equal Surplus Sharing Approaches defined over the grey class. Based on the application data from the multi-owner logistics network example taken from Baykasoğlu and Kubur Özbel (2019), the Grey Banzhaf value, $\mathcal{G}CIS$-value, $\mathcal{G}ENSC$-value and $\mathcal{G}ED$-value have been calculated for both flow and risk.

This study contributes to the literature in several ways. First, the egalitarian distribution approach solutions used by Dönmez et al. (2024) in grey inventory games have been applied to maximum flow games. In doing so, the applicability of grey solutions from the egalitarian distribution approach to various cooperative network problems has been demonstrated. Second, in addition to the Shapley value approach of Baykasoğlu and Kubur Özbel (2019), both flow and risk have been allocated simultaneously under four different fairness criteria. This approach offers decision-makers a spectrum of solutions suitable to their preferences rather than imposing a single solution. Third, the coalition synergy (collaboration producing a risk lower than the sum of individual risks) that emerged in the CIS-value analysis of the grey risk game has been demonstrated within the framework of grey uncertainty.

From the perspective of various application areas, these findings offer concrete guidance to decision-makers operating multi-owner logistics networks (Frisk et al., 2010), natural gas transmission systems (Koch et al., 2015; Tran et al., 2018) and electrical transmission grids (Banez-Chicharro et al., 2017). In particular, the fact that the ENSC-value lies within the core indicates that this solution should be preferred in applications where long-term coalition stability is sought (in contexts such as multinational pipeline operations or joint logistics network management).

Several directions are suggested for future research. First and foremost, heterogeneous-aspiration-level scenarios, in which owners with different risk preferences form coalitions, can be investigated. Furthermore, flow loss can be incorporated into the model (Olgun & Aydemir, 2021), the nucleolus and the başlat denklem tau-value adapted to grey maximum flow games, and a systematic comparison can be conducted between the grey Shapley value and the egalitarian distribution approach solutions proposed in this study.

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