Chapter 4

Extensions of Singular Fourth Order Dynamic Operators with Transmission Conditions

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Abstract

In the present paper, a singular fourth-order dynamic operator on time scales is investigated in the presence of impulsive conditions. Initially, a suitable boundary value framework adapted to the impulsive structure of the operator is constructed. Subsequently, the maximal dissipative, self-adjoint, and further appropriate extensions of the associated singular fourth-order differential operator, which acts on unbounded time scales under impulsive effects, are comprehensively characterized.

1. Introduction

The calculus of time scales provides a comprehensive framework for the unification of continuous and discrete mathematical paradigms. Since its formal conceptualization in the early 1990s (Hilger,1990), this field has emerged as a pivotal area of research, effectively bridging the theoretical gap between differential and difference equations. The resulting theory of dynamic equations offers a robust analytical toolset for modeling complex phenomena across diverse scientific domains. Notably, time scale analysis has become indispensable in simulating heat conduction, entomological population shifts, infectious disease trajectories, financial market volatility, and the computational dynamics of neural networks (Hilger,1990; Thomas et al., 2005).

Furthermore, the expansion of symmetric operators is an essential element in diverse sectors of mathematical physics, notably within quantization issues and solvable quantum mechanics paradigms. Initially established by J. von Neumann (Neumann, 1929), the theory of extensions was significantly refined by Calkin (Calkin, 1939), who characterized self-adjoint extensions through abstract boundary conditions. Subsequent advancements by Rofe-Beketov (Rofe-Beketov,1969) introduced the use of linear relations to delineate these extensions. The establishment of boundary value space theory by Bruk (Bruk, 1976) and Kochubei (Kochubei, 1975) constituted a significant breakthrough, offering a systematic framework to classify all maximal dissipative, accretive,

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https://doi.org/10.58830/ozgur.pub1361.c5494

and self-adjoint extensions of symmetric operators. This subject remains a focal point for mathematical research (Allahverdiev, 1995; Tuna and Allahverdiev, 2018), with exhaustive theoretical treatments available in (Gorbachuk et al., 1989)

On the other hand, impulsive problems have attracted substantial academic attention, resulting in an extensive body of both theoretical and empirical research (see (Allahverdiev et al., 2023; Allahverdiev and Tuna, 2019; Aydemir et al., 2018; Aydemir et al., 2019; Amirov et al., 2021; Güldü, 2013). These phenomena are intrinsic to numerous scientific fields characterized by abrupt transitions, such as heat and mass transfer mechanisms (Lykov and Mikhailov, 1965), geophysical modeling (Lapwood and Usami, 1981) and the intricacies of radio science (Litvinenko and Soshnikov, 1964). The comprehensive literature surrounding these issues underscores their vital role in precisely modeling systems that experience instantaneous perturbations.

The primary objective of the present study is oriented toward exploring a singular fourth-order dynamic operator on time scales, with particular consideration given to the effects of impulsive perturbations. By integrating impulsive perturbations into the time scale framework, we provide a rigorous analysis of the operator's structural properties. This includes the systematic construction of boundary value spaces and the subsequent classification of its fundamental extensions.

The structure of this paper is organized as follows: Section 2 provides a concise overview of the fundamental properties of time scales to establish a necessary theoretical background. In Section 3, we establish a boundary value space associated with singular fourth-order dynamic operators, focusing exclusively on the Limit-2 case. Within this framework, we offer a comprehensive characterization of all maximal dissipative, accretive, and self-adjoint extensions among others utilizing explicit boundary conditions. Finally, Section 4 extends this analysis to the Lim-4 case, providing an exhaustive description of all operator extensions under these conditions.

2. Preliminaries

This section briefly outlines the fundamental tenets of time scale calculus requisite for the ensuing analysis. For a comprehensive theoretical background and rigorous derivations, we refer the reader to the established literature (Bohner and Peterson, 2001; Bohner and Peterson, 2003; Atici and Guseinov, 2002; Lakshmikantham et al.,2013; Guseinov, 2005; Hilger, 1990; Huseynov, 2012).

Definition 2.1.: Consider a time scale $\mathbb{T}$. The mapping $\sigma: \mathbb{T} \to \mathbb{T}$, designated as the forward jump operator, is formulated as:

$$\sigma(r) = \inf\{w \in \mathbb{T}: w > r\},\ r \in \mathbb{T}.$$

Correspondingly, the backward jump operator $\rho: \mathbb{T} \to \mathbb{T}$ is defined by:

$$\rho(r) = \sup\{w \in \mathbb{T}: w < r\},\ r \in \mathbb{T}.$$

To quantify the step size between points, we introduce the graininess functions $\mu_\sigma(r) = \sigma(r) - r$ and $\mu_\rho(r) = \rho(r) - r$. The topological classification of any point $r \in \mathbb{T}$ is determined by these functions: r is categorized as left-scattered if $\mu_\rho(r) \neq 0$ and left-dense if $\mu_\rho(r) = 0$. Analogously, it is right-scattered if $\mu_\sigma(r) \neq 0$ and right-dense if $\mu_\sigma(r) = 0$. Furthermore, we define the restricted sets $\mathbb{T}^k$, $\mathbb{T}_k$,and $\mathbb{T}^*$ to ensure the existence of derivatives at boundary points. Specifically, $\mathbb{T}^k$ is obtained by removing the maximum element of $\mathbb{T}$ if it is left-scattered; otherwise, $\mathbb{T}^k = \mathbb{T}$. In a symmetric fashion, $\mathbb{T}_k$ excludes the minimum element if it is right-scattered. The intersection of these subsets is denoted as $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

Definition 2.2.: A function f defined on $\mathbb{T}$ is considered Δ -differentiable at $r \in \mathbb{T}$ provided there exists a value $f^\Delta(r)$ satisfying the condition that for any $\varepsilon > 0$, a neighborhood U of r exists such that for all $w \in U$:

$$|f(\sigma(r)) - f(w) - f^\Delta(r)(\sigma(r) - w)| \leq \varepsilon|\sigma(r) - w|.$$

By utilizing the backward jump operator ρ, the dual concept of ∇-differentiability is defined in an analogous fashion. Furthermore, within the domain of continuously differentiable functions, the following dual relations establish the connection between these two derivative operators (Bohner and Peterson, 2001):

$$f^\Delta(r) = f^\nabla(\sigma(r)), \qquad f^\nabla(r) = f^\Delta(\rho(r)).$$

Example 2.1.: The following instances demonstrate how the Δ-differential operator generalizes classical calculus:

When $\mathbb{T} = \mathbb{R}$, the forward jump operator becomes $\sigma(r) = r$, and the Δ-derivative simplifies to the traditional derivative $f^\Delta(r) = f'(r)$.

When $\mathbb{T} = \mathbb{Z}$, we have $\sigma(r) = r + 1$, which identifies the operator with the forward difference $f^\Delta(r) = \Delta f(r) = f(r+1) - f(r)$.

When $\mathbb{T} = q^{\mathbb{N}_0}$, the jump operator is defined as $\sigma(r) = qr$ for $q > 1$, resulting in the q-difference quotient $f^\Delta(r) = \frac{f(qr)-f(r)}{qr-r}$.

Definition 2.3.: Let f denote a function defined on the time scale $\mathbb{T}$, and suppose that $k, m \in \mathbb{T}$. We designate $F: \mathbb{T} \to \mathbb{R}$ as a Δ-antiderivative of f if the equality $F^\Delta(r) = f(r)$ is satisfied for every $r \in \mathbb{T}^k$. In this framework, the definite integral of f is determined by the fundamental relation:

$$\int_k^m f(r)\Delta r = F(m) - F(k)\,.$$

A parallel construction is utilized to define the ∇-antiderivative, substituting the forward difference mechanism with its backward counterpart.

The space $L^2_\Delta(\mathbb{T}^*)$ comprises all functions defined on $\mathbb{T}^*$ for which the following norm condition is satisfied:

$$\|f\| := \left(\int_k^l |f(r)|^2 \Delta r\right)^{1/2} + \left(\int_l^m |f(r)|^2 \Delta r\right)^{1/2} < \infty.$$

Assume $\mathbb{T}$ denotes a time scale which is bounded below but extends indefinitely upward, satisfying $\inf\mathbb{T} = k > -\infty$ and $\sup\mathbb{T} = \infty$. This domain shall also be denoted as $[k, l) \cup (l, \infty)_{\mathbb{T}}$.

The set $L^2_\Delta[k, l) \cup (l, \infty)_{\mathbb{T}}$ forms a Hilbert space when endowed with the inner product given by:

$$\langle f, g\rangle := \int_k^l f(r)\overline{g(r)}\Delta r + \int_l^\infty f(r)\overline{g(r)}\Delta r, \quad f, g \in L^2_\Delta[k, l) \cup (l, \infty)_{\mathbb{T}}.$$

(see (Rynne, 2007)).

The following fourth-order dynamic equation is examined within this framework:

$$\Upsilon\varsigma(r) := (p_0\varsigma^{\Delta\nabla})^{\nabla\Delta}(r) - (p_1\varsigma^{\nabla})^{\Delta} + p_2(r)\varsigma(r) = \lambda\varsigma(r), r \in [k, l) \cup (l, \infty)_{\mathbb{T}}, \qquad (2.1)$$

the analysis assumes that the coefficients p_0, p_1 and p_2 are real-valued. Additionally, p_0^{-1}, p_1 and p_2 are required to be locally Δ-integrable on $[k, l) \cup (l, \infty)_{\mathbb{T}}$, with the further stipulation that $p_0 > 0$ holds throughout the domain $[k, l) \cup (l, \infty)_{\mathbb{T}}$.

A similar problem has also been investigated without impulsive conditions (Tuna and Bayrak, 2018; Tuna and Bulut, 2018).

To streamline the notation, the following shorthand will be employed:

$$\varsigma^{[0]} = \varsigma$$
$$\varsigma^{[1]} = \varsigma^{\Delta}$$
$$\varsigma^{[2]} = p_0\varsigma^{\Delta\nabla}$$
$$\varsigma^{[3]} = p_1\varsigma^{\nabla} - \left(\varsigma^{[2]}\right)^{\nabla}$$
$$\varsigma^{[4]} = p_2\varsigma - \left(\varsigma^{[3]}\right)^{\Delta}.$$

Subsequently, we transform the dynamic equation given in (2.1) into its equivalent Hamiltonian system representation. To this end, we introduce the following vector-valued variables:

$$X = \begin{bmatrix} \varsigma \\ \varsigma^{\Delta} \\ -\left(p_0\varsigma^{\Delta\nabla}\right)^{\Delta} + p_1\varsigma^{\Delta} \\ p_0\varsigma^{\Delta\Delta} \end{bmatrix}, \quad \hat{X} = \begin{bmatrix} \varsigma \\ \varsigma^{\Delta} \\ -\left(p_0\varsigma^{\Delta\nabla}\right)^{\nabla} + p_1\varsigma^{\nabla} \\ p_0\varsigma^{\Delta\nabla} \end{bmatrix}$$

which transforms (2.1) into the following form:

$$J\,\hat{X}^{\Delta} = (\lambda C + E)X, \qquad (2.2)$$

where

$$C=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\ E=\begin{bmatrix}-p_2&0&0&0\\0&-p_1&1&0\\0&1&0&0\\0&0&0&1/p_0\end{bmatrix}$$

and

$$J=\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix}.$$

It should be noted that C and E possess real and symmetric properties.

The Green identity corresponding to the solutions $\varsigma(r,\lambda)$ and $z(r,\lambda)$ takes the form

$$\int_k^l (\Upsilon\varsigma)(r)\,\overline{z(r)}\Delta r+\int_l^\infty (\Upsilon\varsigma)\,(r)\overline{z(r)}\Delta r-\int_k^l \varsigma(r)\,\overline{(\Upsilon z)(r)}\Delta r-\int_l^\infty \varsigma\,(r)\overline{(\Upsilon z)(r)}\Delta r$$

$$=[\varsigma,z]_\infty-[\varsigma,z](l+)+[\varsigma,z](l-)-[\varsigma,z]_k], \tag{2.3}$$

here $[\varsigma,z]_r:=\varsigma^{[0]}(r)\overline{z}^{[3]}(r)-\varsigma^{[3]}(r)\overline{z}^{[0]}(r)+\varsigma^{[1]}(r)\overline{z}^{[2]}(r)-\varsigma^{[2]}(r)\overline{z}^{[1]}(r)$ and $[\varsigma,z]_\infty:=\lim_{r\to\infty}[\varsigma,z]_r$ (see (Anderson et al., 2006). It follows that the limit $[\varsigma,z]_\infty$ exists and remains finite. Denoting by $X(r,\lambda)$ and $Z(r,\lambda)$ the associated vector functions introduced in (2.2), one obtains

$$X^TJZ(r)=[\varsigma,z]_r.$$

Let $\tau_i, 1\le i\le 4$, represent the solutions of Eq. (2.1) determined under the normalization requirement

$$p_0^2(r)W(\tau_1,\tau_2,\tau_3,\tau_4)=1$$

and

$$\begin{bmatrix}[\tau_1,\tau_1]&[\tau_2,\tau_1]&[\tau_3,\tau_1]&[\tau_4,\tau_1]\\ [\tau_1,\tau_2]&[\tau_2,\tau_2]&[\tau_3,\tau_2]&[\tau_4,\tau_2]\\ [\tau_1,\tau_3]&[\tau_2,\tau_3]&[\tau_3,\tau_3]&[\tau_4,\tau_3]\\ [\tau_1,\tau_4]&[\tau_2,\tau_4]&[\tau_3,\tau_4]&[\tau_4,\tau_4]\end{bmatrix}=\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix},$$

where the Wronskian corresponding to τ_1,τ_2,τ_3 and τ_4 is given by (see (Anderson et al., 2006)

$$W(\tau_1,\tau_2,\tau_3,\tau_4)=\begin{vmatrix}\tau_1&\tau_2&\tau_3&\tau_4\\ \tau_1^{[1]}&\tau_2^{[1]}&\tau_3^{[1]}&\tau_4^{[1]}\\ \tau_1^{[2]}&\tau_2^{[2]}&\tau_3^{[2]}&\tau_4^{[2]}\\ \tau_1^{[3]}&\tau_2^{[3]}&\tau_3^{[3]}&\tau_4^{[3]}\end{vmatrix}.$$

Lemma 2.1.: For arbitrary $f, g \in L^2_\Delta[k,l) \cup (l,\infty)_{\mathbb{T}}$, the following Plücker relation holds:

$$[f,g]_r = \begin{vmatrix} [\tau_2, g]_r & [g,\tau_4]_r \\ [\tau_2, f]_r & [f,\tau_4]_r \end{vmatrix} + \begin{vmatrix} [\tau_1, g]_r & [g,\tau_3]_r \\ [\tau_1, f]_r & [f,\tau_3]_r \end{vmatrix}. \quad (2.4)$$

Proof. The argument proceeds analogously to the proof of Lemma 1 in (Fulton, 1989); therefore, the details are omitted.

We denote by

$$D_{max} = \left\{ \tau \in L^2_\Delta[k,l) \cup (l,\infty)_{\mathbb{T}} \middle| \begin{array}{c} the\ first\ three\ \Delta\ derivatives\ are \\ locally\ \Delta - absolutely\ continuous \\ in\ [k,l) \cup (l,\infty)_{\mathbb{T}}, Z(l+) = UZ(l-) \\ and\ \Upsilon\tau \in L^2_\Delta[k,l) \cup (l,\infty)_{\mathbb{T}}. \end{array} \right\},$$

where

$$Z = \begin{pmatrix} \varsigma^{(0)} \\ \varsigma^{(1)} \\ \varsigma^{(3)} \\ \varsigma^{(2)} \end{pmatrix}, U = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & \frac{1}{m} & 0 \\ 0 & 0 & 0 & \frac{1}{n} \end{pmatrix},$$

and $m, n \in \mathbb{R},\ m, n \neq 0$.

Through the relation $\Gamma_{\max}\tau = \Upsilon\tau$, we characterize the maximal operator $\Gamma_{\max}$ on D_{max} .
The linear manifold D_{min} is designated to consist of all functions $\varsigma \in D_{max}$ that satisfy

$$\varsigma^{[0]}(\mathrm{k}) = \varsigma^{[1]}(\mathrm{k}) = \varsigma^{[2]}(\mathrm{k}) = \varsigma^{[3]}(\mathrm{k}) = 0, = [\varsigma, z]_\infty = 0, \forall z \in D_{max}. \quad (2.5)$$

Restricting the operator operator $\Gamma_{\max}$ to the domain D_{min} yields the minimal operator $\Gamma_{\min}$.

We then have the adjoint relation $\Gamma^*_{\min} = \Gamma_{\max}$. Furthermore, we note that $\Gamma_{\min}$ is a closed symmetric operator characterized by deficiency indices (2,2), (3,3) or (4,4) (see, (Naimark, 1968; Fulton, 1989)).

We now recall the following result.

Definition 2.4.: A linear operator V with dense domain $D(V)$ in a Hilbert space H is said to be dissipative (respectively, accumulative) if $\mathrm{Im}(Vf, f) \geq 0$, (respectively, $\mathrm{Im}(Vf, f) \leq 0$) for every $f \in D(V)$.It is termed maximal dissipative (respectively, maximal accumulative) provided that it admits no proper dissipative (accumulative) extension (see, (Maksudov and Allahverdiev, 1993; Allahverdiev, 2016; Canoğlu and Allahverdiev, 2003)).

Definition 2.5.: Let V be a closed symmetric operator in a Hilbert space H with equal deficiency indices. Then, a triplet $(\mathbb{H}, \gamma_1, \gamma_2)$ is classified as a boundary value space for V, under the condition that γ_1 and γ_2 constitute linear mappings from $D(V^*)$ into H that satisfy the following criteria:

i) For all $f, g \in D(V^*)$ the following relation holds:

$$\langle V^* f, g\rangle_H - \langle f, V^* g\rangle_H = \langle \gamma_1 f, \gamma_2 g\rangle_{\mathbb{H}} - \langle \gamma_2 f, \gamma_1 g\rangle_{\mathbb{H}};$$

ii) Given arbitrary $F_1, F_2 \in H$ there exists an element $f \in D(V^*)$ satisfying $\gamma_1 f = F_1$ and $\gamma_2 f = F_2$ (see (Gorbachuk and Gorbachuk, 1984)).

3. Lim-2 Case

The primary objective of this section is to analyze singular fourth-order dynamic operators under the framework of the Limit-2 case. Within the context of boundary value space theory, we provide a complete characterization of maximal dissipative, accretive, self-adjoint, and other admissible extensions by specifying the corresponding boundary conditions.

Suppose the symmetric operator $\Gamma_{\min}$ is characterized by deficiency indices (2,2), which corresponds to the Limit-2 case. In this situation, the boundary form satisfies $[\tau, z]_\infty = 0$ for every z (see (Naimark, 1968)). Moreover, the domain D_{min} of $\Gamma_{\min}$ is exactly the set of all vectors $\varsigma \in D_{min}$ for which the boundary conditions $\varsigma^{[0]}(k) = \varsigma^{[1]}(k) = \varsigma^{[2]}(k) = \varsigma^{[3]}(k) = 0$ are fulfilled.

We introduce the linear operators S_1 and S_2 mapping D_{max} into $\mathbb{C}^2$ by

$$S_1 f = \begin{pmatrix} -\varsigma^{[0]}(k) \\ \varsigma^{[1]}(k) \end{pmatrix}, \; S_2 f = \begin{pmatrix} \varsigma^{[3]}(k) \\ \varsigma^{[2]}(k) \end{pmatrix}, \tag{3.1}$$

where the components are given in terms of the boundary values at k. We proceed by formulating and establishing several auxiliary lemmas.

Lemma 3.1.: For any $\tau, z \in D_{max}$, the following identity holds:

$$\langle \Gamma_{\max}\varsigma, z\rangle_{L^2_\Delta} - \langle \varsigma, \Gamma_{\max} z\rangle_{L^2_\Delta} = \langle S_1\varsigma, S_2 z\rangle_{\mathbb{C}^2} - \langle S_2\varsigma, S_1 z\rangle_{\mathbb{C}^2}. \tag{3.2}$$

Proof. For all $\varsigma, z \in D_{max}$ Green's identity takes the form

$$\langle \Gamma_{\max}\varsigma, z\rangle_{L^2_\Delta} - \langle \varsigma, \Gamma_{\max} z\rangle_{L^2_\Delta} = -[\varsigma, z]_k. \tag{3.3}$$

Hence, we obtain

$$\begin{aligned} \langle S_1\varsigma, S_2 z\rangle_{\mathbb{C}^2} - \langle S_2\varsigma, S_1 z\rangle_{\mathbb{C}^2} &= -\varsigma^{[0]}(k)\bar{z}^{[3]}(k) - \varsigma^{[1]}(k)\bar{z}^{[2]}(k) \\ &\quad +\varsigma^{[3]}(k)\bar{z}^{[0]}(k) + \varsigma^{[2]}(k)\bar{z}^{[1]}(k) \\ &= -[\varsigma, z]_k\,. \end{aligned}$$

By applying relation (3.3), identity (3.2) follows immediately.

Lemma 3.2.: For arbitrary complex numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ there exists a function $\varsigma \in D_{max}$ such that

$$\varsigma^{[0]}(k) = \alpha_1, \varsigma^{[1]}(k) = \alpha_2, \varsigma^{[2]}(k) = \alpha_3, \varsigma^{[3]}(k) = \alpha_4.$$

Proof. Let $u = \binom{u_1}{u_2}$, and $v = \binom{v_1}{v_2}$ be arbitrary elements of $\in \mathbb{C}^2$. Define the vector-valued function

$$\varsigma(r) = \alpha_1(r)u_1 + \alpha_2(r)v_1 + \alpha_3(r)u_2 + \alpha_4(r)v_2,$$

where $\alpha_i(r) \in L^2_\Delta[k,l) \cup (l,\infty)_{\mathbb{T}}$ $(i = 1,\ldots,4)$, and the functions α_i satisfy the boundary requirements

$$\begin{array}{llll} \alpha_1^{[0]}(k) = 1 & \alpha_1^{[1]}(k) = 0 & \alpha_1^{[2]}(k) = 0 & \alpha_1^{[3]}(k) = 0 \\ \alpha_2^{[0]}(k) = 0 & \alpha_2^{[1]}(k) = 0 & \alpha_2^{[2]}(k) = 0 & \alpha_2^{[3]}(k) = 1 \\ \alpha_3^{[0]}(k) = 0 & \alpha_3^{[1]}(k) = -1 & \alpha_3^{[2]}(k) = 0 & \alpha_3^{[3]}(k) = 0 \\ \alpha_4^{[0]}(k) = 0 & \alpha_4^{[1]}(k) = 0 & \alpha_4^{[2]}(k) = 1 & \alpha_4^{[3]}(k) = 0 \end{array}.$$

Then $\varsigma \in D_{max}$ and the boundary mappings satisfy $S_1\varsigma = u,\ S_2\varsigma = v$.

Therefore, the following statements are valid.

Theorem 3.1.: As formulated in (3.1), the triplet $\langle \mathbb{C}^2, S_1, S_2 \rangle$, constitutes a boundary value space corresponding to the operator $\Gamma_{\min}$.

Corollary 3.1.: Assume that M is a contraction acting in $\mathbb{C}^2$. Consider the operator obtained by restricting $\Gamma_{\min}$ to the class of functions $\varsigma \in D_{max}$ for which either

$$(M - I)S_1\varsigma + i(M + I)S_2\varsigma = 0 \tag{3.4}$$

or

$$(M - I)S_1\varsigma - i(M + I)S_2\varsigma = 0 \tag{3.5}$$

holds. Enforcing these criteria results, respectively, in the maximal dissipative extension and the maximal accretive extension of the operator $\Gamma_{\min}$. On the other hand, any maximal dissipative (respectively, maximal accretive) extension of $\Gamma_{\min}$ can be realized as the restriction of $\Gamma_{\max}$ to the collection of functions $\varsigma \in D_{max}$ that satisfy condition (3.4) (respectively, (3.5)), and this extension uniquely specifies the contraction M. Moreover, when M is an isometry, conditions (3.4)–(3.5) characterize the maximal symmetric extensions of $\Gamma_{\min}$ in $L^2_\Delta[k,l) \cup (l,\infty)_{\mathbb{T}}$. If, in addition, M is a unitary operator, then these criteria determine the self-adjoint extensions.

In this context, we consider the following boundary conditions

$$\varsigma^{[3]}(k) - h_1\varsigma^{[0]}(k) = 0,$$

$$\varsigma^{[1]}(k) - h_2\varsigma^{[2]}(k) = 0,$$

for $\mathrm{Im}h_1 \geq 0$ or $h_1 = \infty$ and $\mathrm{Im}h_2 \geq 0$ or $h_2 = \infty$, the above separated boundary conditions generate maximal dissipative extensions of $\Gamma_{\min}$; if $\mathrm{Im}h_1 = 0$ or $\mathrm{Im}h_2 = 0$, (or $h_1, h_2 = \infty$), they yield self-adjoint extensions.

4. Lim-4 Case

The aim of this part of the study is to analyze singular fourth-order dynamic operators within the framework of the Limit-4 case. Through the formulation of proper boundary conditions, a comprehensive classification of their maximal dissipative, maximal accretive, self-adjoint, and further allowable extensions is presented.

Suppose that the operator $\Gamma_{\min}$ is characterized by deficiency indices (4,4). Consequently, the functions τ_i lie in the domain D_{max} and satisfy $\tau_i \in L_\Delta^2[k,l) \cup (l,\infty)_{\mathbb{T}}$ for every $i \in \{12,3,4\}$.

Theorem 4.1.: Definition domain D_{min} associated with the operator $\Gamma_{\min}$ comprises exactly those functions $\varsigma \in D_{max}$ for which the following requirements are fulfilled:

$$\begin{gathered}\varsigma^{[0]}(\mathrm{k}) = \varsigma^{[1]}(\mathrm{k}) = \varsigma^{[2]}(\mathrm{k}) = \varsigma^{[3]}(\mathrm{k}) = 0,\\ [\varsigma,\tau_1]_\infty = [\varsigma,\tau_2]_\infty = [\varsigma,\tau_3]_\infty = [\varsigma,\tau_4]_\infty = 0\end{gathered} \tag{4.1}$$

Proof. The required conclusion follows directly from relations ([23]) and ([p]).

Define the linear operator Ω_1 and Ω_2 from D_{max} into $\mathbb{C}^4$ by

$$\Omega_1\varsigma = \begin{pmatrix} -\varsigma^{[0]}(k) \\ -\varsigma^{[1]}(k) \\ [\varsigma,\tau_2]_\infty \\ [\varsigma,\tau_1]_\infty \end{pmatrix}, \Omega_2\tau = \begin{pmatrix} \varsigma^{[3]}(k) \\ \varsigma^{[2]}(k) \\ [\varsigma,\tau_4]_\infty \\ [\varsigma,\tau_3]_\infty \end{pmatrix}. \tag{4.2}$$

Afterward, the following are obtained.

Theorem 4.2.: For the operator $\Gamma_{\min}$, we establish that the triple $\langle \mathbb{C}^4, \Omega_1, \Omega_2 \rangle$ given in (4.2) constitutes a boundary value space.

Proof. Given any $\varsigma, z \in D_{max}$, it follows that

$$\begin{aligned}\langle\Omega_1\varsigma,\Omega_2 z\rangle-(\langle\Omega_2\varsigma,\Omega_1 z\rangle &= -\varsigma^{[0]}(k)\bar{z}^{[3]}(k)-\varsigma^{[1]}(k)\bar{z}^{[2]}(k)\\ &+[\varsigma,\tau_2]_\infty[z,\tau_4]_\infty+[\varsigma,\tau_1]_\infty[z,\tau_3]_\infty\\ &+\varsigma^{[3]}(k)\overline{z}^{[0]}(k)+\varsigma^{[2]}(k)\bar{z}^{[1]}(k)\\ &-[\varsigma,\tau_4]_\infty[z,\tau_2]_\infty-[\varsigma,\tau_3]_\infty[z,\tau_1]_\infty\\ &= -\varsigma^{[0]}(k)\bar{z}^{[3]}(k)-\varsigma^{[1]}(k)\bar{z}^{[2]}(k)\\ &+\varsigma^{[3]}(k)\bar{z}^{[0]}(k)+\varsigma^{[2]}(k)\bar{z}^{[1]}(k)\\ &+[\varsigma,\tau_4]_\infty[\tau_2,z]_\infty-[\varsigma,\tau_2]_\infty[\tau_4,z]_\infty\\ &+[\varsigma,\tau_3]_\infty[\tau_1,z]_\infty-[\tau_1,\varsigma]_\infty[z,\tau_3]_\infty.\end{aligned}$$

By the Plücker identity (2.4), we obtain

$$\langle\Omega_1\varsigma,\Omega_2 z\rangle-(\langle\Omega_2\varsigma,\Omega_1 z\rangle=[\varsigma,z]_\infty-[\varsigma,z]_k.$$

Using Green's formula (2.3), we obtain

$$\langle\Omega_1\varsigma,\Omega_2 z\rangle-(\langle\Omega_2\varsigma,\Omega_1 z\rangle=\langle\Gamma_{\max}\varsigma,z\rangle_{L^2_\Delta}-\langle\varsigma,\Gamma_{\max}z\rangle_{L^2_\Delta},$$

which shows that the first requirement in the definition of a boundary value space is satisfied.

The verification of the second condition is provided by the lemma stated below.

Lemma 4.1.: For any complex numbers $\alpha_0,\alpha_1,\alpha_2,\alpha_3,\beta_0,\beta_1,\beta_2$,and β_3 there is a function $\varsigma\in D_{max}$ satisfying

$$\begin{aligned}\varsigma^{[0]}(k) &= \alpha_0,\ \varsigma^{[1]}(k)=\alpha_1,\ \varsigma^{[2]}(k)=\alpha_2,\ \varsigma^{[3]}(k)=\alpha_3\\ [\varsigma,\tau_1]_\infty &= \beta_0,\ [\varsigma,\tau_2]_\infty=\beta_1,\ [\varsigma,\tau_3]_\infty=\beta_2,\ [\varsigma,\tau_4]_\infty=\beta_3\ .\end{aligned}$$

Proof. Let f be any function belonging to $L^2_\Delta[k,l)\cup(l,\infty)_\mathbb{T}$ such that

$$\begin{aligned}&\langle f,\tau_1\rangle_{L^2_\Delta}=\beta_0+\alpha_3,\langle f,\tau_2\rangle_{L^2_\Delta}=\beta_1-\alpha_1\\ &\langle f,\tau_3\rangle_{L^2_\Delta}=\beta_2-\alpha_0,\langle f,\tau_4\rangle_{L^2_\Delta}=\beta_3+\alpha_2.\end{aligned}\tag{4.3}$$

Such a function f exists, and it can be chosen as a linear combination of τ_1,τ_2,τ_3 and τ_4. In fact, let

$$f=c_1\tau_1+c_2\tau_2+c_3\tau_3+c_4\tau_4.$$

Then the relations in (4.3) form a linear system for the coefficients c_1,c_2,c_3,c_4. The determinant of this system coincides with the Gram determinant associated with linearly independent functions τ_1,τ_2,τ_3 τ_4 , and hence it is nonzero.

Denote by $\varsigma(r)$ a function satisfying the equation $\Upsilon(\varsigma) = f$ which fulfills the initial conditions

$$\varsigma^{[0]}(k) = \alpha_0,\ \varsigma^{[1]}(k) = \alpha_1,\ \varsigma^{[2]}(k) = \alpha_2,\ \varsigma^{[3]}(k) = \alpha_3.$$

Suppose that $\varsigma(r)$ represents the required element. Applying Green's formula (2.3) to the pair $\varsigma(r)$ and τ_j, we obtain

$$\langle f, \tau_j\rangle_{L^2_\triangle} = \langle \Upsilon(\varsigma), \tau_j\rangle_{L^2_\triangle} = [\varsigma, \tau_j]_\infty - [\varsigma, \tau_j]_0,\ j = 1, \ldots, 4.$$

Since $\Upsilon(\tau_j) = 0$ for $(j = 1,2,3,4)$ and the initial data $\varsigma^{[0]}(k) = \alpha_0$, $\varsigma^{[1]}(k) = \alpha_1$, $\varsigma^{[2]}(k) = \alpha_2$, $\varsigma^{[3]}(k) = \alpha_3$, hold, it follows that

$$[\varsigma, \tau_j]_k = \begin{cases} -\alpha_3, & \text{when } j = 1 \\ \alpha_1, & \text{when } j = 2 \\ \alpha_0, & \text{when } j = 3 \\ -\alpha_2, & \text{when } j = 4 \end{cases}.$$

Thus we get

$$\begin{aligned}
\langle f, \tau_1\rangle_{L^2_\triangle} &= [\varsigma, \tau_1]_\infty - [\varsigma, \tau_1]_k \\
&= [\varsigma, \tau_1]_\infty + \alpha_3, \\
\langle f, \tau_2\rangle_{L^2_\triangle} &= [\varsigma, \tau_2]_\infty - [\varsigma, \tau_2]_k \\
&= [\varsigma, \tau_2]_\infty - \alpha_1, \\
\langle f, \tau_3\rangle_{L^2_\triangle} &= [\varsigma, \tau_3]_\infty - [\varsigma, \tau_3]_k \\
&= [\varsigma, \tau_3]_\infty - \alpha_0, \\
\langle f, \tau_4\rangle_{L^2_\triangle} &= [\varsigma, \tau_4]_\infty - [\varsigma, \tau_4]_k \\
&= [\varsigma, \tau_4]_\infty + \alpha_2.
\end{aligned}$$

According to (4.3), we have

$$[\varsigma, \tau_1]_\infty = \beta_0,\ [\varsigma, \tau_2]_\infty = \beta_1,\ [\varsigma, \tau_3]_\infty = \beta_2,\ [\varsigma, \tau_4]_\infty = \beta_3.$$

Corollary 4.1.: Let M be a contraction operator on $\mathbb{C}^4$. We consider the operator defined by restricting $\Gamma_{\min}$ to the class of functions $\varsigma \in D_{max}$ that satisfy either the boundary condition

$$(M - I)\Omega_1\varsigma + i(M + I)\Omega_2\varsigma = 0 \tag{4.4}$$

or

$$(M - I)\Omega_1\varsigma - i(M + I)\Omega_2\varsigma = 0. \tag{4.5}$$

Accordingly, these operators represent the maximal dissipative and maximal accretive extensions of $\Gamma_{\min}$, respectively. Conversely, any maximal dissipative (respectively, accretive) expansion of $\Gamma_{\min}$ is achievable through the restriction of $\Gamma_{\max}$ to those functions $\varsigma \in D_{max}$ satisfying condition (4.4) (respectively, (4.5)), with this expansion uniquely fixing the contraction M.

Moreover, when M is an isometric operator, conditions (4.4) and (4.5) characterize the maximal symmetric extensions of $\Gamma_{\min}$ in $L^2_\Delta[k,l) \cup (l,\infty)_{\mathbb{T}}$. In the case where M is unitary property , these boundary conditions correspond to self-adjoint extensions.

In this connection, we take into account the following boundary conditions:

$$\varsigma^{[3]}(k) - h_1 \varsigma^{[0]}(k) = 0$$

$$\varsigma^{[2]}(k) - h_2\, \varsigma^{[1]}(k) = 0,$$

$$[\varsigma, \tau_4]_\infty - h_3[\varsigma, \tau_2]_\infty = 0,$$

$$[\varsigma, \tau_3]_\infty - h_4[\varsigma, \tau_1]_\infty = 0,$$

under separated boundary conditions, the maximal dissipative (self-adjoint) extensions of the operator $\Gamma_{\min}$are uniquely determined by the constraints $\mathrm{Im} h_i \geq 0$ or $h_i = \infty$ (specifically $\mathrm{Im} h_i = 0$ or $h_i = \infty$) for each index $i \in \{1,2,3,4\}$. These parameters define the precise domain boundaries required for the validity of such extensions.

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