Chapter 5

Embeddability Degree of Finite Groups and Approximate Embeddings

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Abstract

This study introduces the *embeddability degree*, a new probabilistic invariant for finite groups motivated by the idea of measuring how closely one group can be embedded into another when an exact embedding does not exist. This proposed invariant provides a quantitative way to evaluate injective maps according to how frequently they preserve the group operation. We establish several fundamental properties of this notion, including a characterization of the case where the embeddability degree is equal to one, a criterion for its vanishing, a symmetry result for groups of equal order, and a composition-type inequality involving three finite groups. We also compute the invariant explicitly for several small cyclic groups, showing that even in elementary cases it reflects meaningful arithmetic and structural features. These initial results suggest that the embeddability degree offers a natural quantitative perspective on classical embedding problems in finite group theory.

1. Introduction

In finite group theory, structural questions have traditionally been studied in an exact algebraic framework. A group is either abelian or nonabelian, a map either preserves the group operation, or it does not, and one group either embeds into another or fails to do so. This classical viewpoint remains central to algebra. However, over the past several decades, probabilistic and quantitative perspectives have introduced a complementary methodology for studying algebraic structures. Rather than treating structural properties as purely binary phenomena, one may ask how frequently a given property occurs or to what extent a function approximates a prescribed algebraic behavior. This perspective has generated a rich body of numerical invariants that provide meaningful structural information about finite groups.

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One of the earliest and most influential examples of this philosophy is the commutativity degree of a finite group, which measures the probability that two randomly selected elements commute. Foundational contributions by Gallagher (1970), Gustafson (1973), and Rusin (1979) demonstrated that probabilistic quantities can encode significant algebraic information regarding finite groups. Later developments by Lescot (1995, 2001) deepened this theory by connecting commutativity probabilities with structural classifications and central extensions. These studies established that probabilistic invariants are not merely numerical curiosities, but rather effective tools for detecting structural properties.

The concept was subsequently generalized in several directions. The relative commutativity degree, introduced and studied in detail by Erfanian et al. (2007) and further examined by Rezaei and Erfanian (2014), provides a refined measure of commutative behavior between a subgroup and the ambient group. Additional extensions include subgroup commutativity measures (Tarnauceanu, 2009), generalized commutativity degrees (Nath & Das, 2011), lower bound analyses (Nath & Das, 2010), and structural classifications involving prescribed commutativity values (Barzegar et al., 2013). More recently, probabilistic commutativity concepts have been extended beyond classical groups. For example, analogous invariants have been studied for semigroups (Ghaneei & Azadi, 2021), higher-order commutativity settings (Hashemi & Pirzadeh, 2022), while crossed module actions (Cetin \& Gurdal, 2024) provide a natural setting for similar quantitative extensions. These developments illustrate the remarkable flexibility of probabilistic algebraic methods.

A related direction concerns algebraic structures arising from operator theory and associated algebraic systems. Although seemingly distinct from finite group theory, quantitative algebraic viewpoints have also appeared in operator-theoretic contexts. For example, Gurdal (2009a, 2009b) investigated structural properties of extended eigenvalues and eigenvectors for classical operators on Wiener-type algebras. Likewise, Chashiani and Rezaei (2021) studied commutativity degree in the setting of group algebras, thereby explicitly linking probabilistic group-theoretic invariants with algebraic operator frameworks. Such interactions suggest that quantitative algebraic concepts often transcend their original settings and admit broader mathematical interpretations.

Motivated by this growing probabilistic perspective, one may naturally shift attention from algebraic properties of groups themselves to the behavior of functions between groups. Classical group theory primarily emphasizes homomorphisms, since these preserve multiplication exactly. Nevertheless, from a quantitative viewpoint, exact preservation may be unnecessarily restrictive. A function may fail to be a homomorphism globally while still behaving homomorphically on a substantial proportion of ordered pairs. This idea motivates the notion of homomorphism degree, which measures the proportion of pairs satisfying the homomorphic identity. Such a concept provides a quantitative measure of approximate algebraic compatibility.

Similarly, surjectivity can also be quantified. Rather than asking whether a homomorphism is surjective in strict sense, one may measure the relative size of its image within the codomain. This leads naturally to a surjectivity degree, offering another numerical invariant describing structural efficiency. Recent work by Uc (2025) investigated quantitative relationships between commutativity, homomorphic behavior, and surjectivity in finite groups, demonstrating that these probabilistic invariants are not isolated notions but rather parts of a coherent quantitative framework.

These observations suggest a broader guiding principle: classical structural properties may often admit meaningful quantitative analogues. The present chapter develops this philosophy in the context of group embeddings.

Recall that a finite group G embeds into a finite group H if there exists an injective group homomorphism $\varphi: G \to H$. From the classical viewpoint, embeddability is a purely binary notion: either such a map exists or it does not. However, this strict dichotomy conceals potentially informative intermediate behavior. Even when no genuine embedding exists, there may still be injective functions from G into H that preserve multiplication on a substantial subset of ordered pairs. Such maps are not homomorphisms in the classical sense, yet they may exhibit significant algebraic compatibility.

This observation motivates the introduction of a new probabilistic invariant, called the *embeddability degree*. Informally, these invariant measures the maximal homomorphic behavior achievable among injective maps between two finite groups. Thus, it combines two distinct structural requirements: injectivity, reflecting the classical embedding condition, and approximate homomorphic preservation, reflecting the probabilistic philosophy of modern quantitative group theory.

The embeddability degree naturally interpolates between exact and approximate structural compatibility. If its value is equal to one, then an actual embedding exists and the classical notion is recovered. At the opposite extreme, if the source group has strictly larger order than the target group, injective maps cannot exist at all, forcing the invariant to vanish. Intermediate values quantify varying levels of approximate compatibility. Consequently, the invariant does not merely indicate whether an embedding exists, but also measures how closely such an embedding can be approximated when exact realization is impossible.

This perspective is conceptually consistent with the broader development of probabilistic methods in algebra, where exact structural properties are complemented by quantitative measurements reflecting partial or approximate behavior. In the setting of finite groups, such approaches have proved particularly fruitful in the study of commutativity-related invariants and their generalizations. In this sense, the embeddability degree may be viewed as a natural continuation of this quantitative line of investigation.

Another appealing aspect of the embeddability degree is its connection with classical structural invariants. Since its definition is built upon approximate homomorphic behavior, it is closely related to homomorphism-based probabilistic measures. At the same time, because injectivity is imposed as an essential condition, the invariant remains fundamentally tied to subgroup structure, group order, and classical embedding constraints. In this sense, it occupies an intermediate position between probabilistic group theory and classical structural algebra.

Even elementary examples suggest that the invariant captures genuinely nontrivial information. For cyclic groups, the interaction between arithmetic divisibility, injectivity constraints, and approximate homomorphic preservation leads to subtle behavior. This indicates that the invariant is sensitive not only to cardinality considerations but also to deeper algebraic structure.

Although the present work focuses primarily on foundational theory, the notion appears sufficiently flexible to admit broader generalizations. One might envision analogous invariants for algebraic systems beyond groups, including semigroup actions, module-theoretic structures, or operator-induced algebraic frameworks. The flexibility of probabilistic and quantitative viewpoints suggests that similar methodological ideas may find applications in broader algebraic settings beyond the finite groups considered here. Therefore, the applicability of quantitative perspectives across diverse mathematical settings, from probabilistic group theory to algebraic operator systems and even fractal-inspired frameworks (Gurdal et al., 2025; Allahverdiev et al., 2026), suggests that the underlying methodology may have substantial conceptual reach.

The main purpose of this chapter is to establish the initial theory of the embeddability degree for finite groups. After recalling the necessary preliminaries, we formally introduce the invariant and investigate its first fundamental properties. In particular, we characterize the extremal case in which the invariant attains the value one, establish vanishing criteria, derive structural symmetry results under equal-order assumptions, and prove composition-type inequalities governing interactions among multiple finite groups. Several explicit computations are also presented for finite cyclic groups in order to illustrate the behavior of the invariant concretely.

We hope that this quantitative viewpoint enriches the classical theory of embeddings by introducing a flexible framework capable of measuring approximate structural compatibility between finite groups.

2. Preliminaries

Throughout this chapter, all groups are assumed to be finite unless otherwise stated. In this section, we recall the basic notions and probabilistic invariants that will be used in the subsequent development.

Let G be a finite group. The identity element of G will be denoted by e_G, or simply by e whenever no ambiguity arises. The center of G is defined by

$$Z(G) := \{x \in G : xg = gx \text{ for all } g \in G\},$$

and for any element $x \in G$, its centralizer is given by

$$C_G(x) := \{g \in G: gx = xg\}.$$

For finite groups G and H, an embedding of G into H means an injective group homomorphism $\varphi: G \to H$.

One of the most classical probabilistic invariants in finite group theory is the commutativity degree, which measures the likelihood that two randomly selected group elements commute.

Definition 2.1.: Let G be a finite group. The *commutativity degree* of G is defined by

$$d(G) := \frac{|\{(x,y) \in G \times G: xy = yx\}|}{|G|^2}. \tag{2.1}$$

This invariant admits an equivalent and often useful formulation in terms of centralizers. Indeed, for each fixed element $x \in G$, the number of elements commuting with x is exactly $|C_G(x)|$. Hence,

$$d(G) = \frac{1}{|G|^2} \sum_{x \in G} |C_G(x)|. \tag{2.2}$$

The commutativity degree satisfies $0 < d(G) \leq 1$; and $d(G) = 1$ if and only if G is abelian.

A natural extension of this notion is the relative commutativity degree, which measures the interaction between a subgroup and the ambient group.

Definition 2.2.: Let G be a finite group and let $H \leq G$. The *relative commutativity degree* of H in G is defined by

$$d(H,G) := \frac{|\{(h,g) \in H \times G: hg = gh\}|}{|H| \cdot |G|}. \tag{2.3}$$

Equivalently, by counting commuting partners via centralizers, we obtain

$$d(H,G) = \frac{1}{|H||G|} \sum_{h \in H} |C_G(h)|. \tag{2.4}$$

These invariant measures the probability that a randomly chosen element of H commutes with a randomly chosen element of G. In particular, $d(G,G) = d(G)$.

To quantify how closely an arbitrary function behaves like a group homomorphism, the homomorphism degree was introduced.

Definition 2.3.: Let G and H be finite groups, and let $f: G \to H$ be an arbitrary function. The *homomorphism degree* of f is defined by

$$\chi(f) := \frac{|\{(a,b) \in G \times G: f(ab) = f(a)f(b)\}|}{|G|^2}. \tag{2.5}$$

Thus, $\chi(f)$ measures the proportion of ordered pairs for which the homomorphism identity holds. Clearly, $0 \leq \chi(f) \leq 1$. Moreover, $\chi(f) = 1$ if and only if f is a group homomorphism.

Another probabilistic invariant relevant to mappings between finite groups is the surjectivity degree.

Definition 2.4.: Let $\varphi: G \to H$ be a group homomorphism between finite groups. The *surjectivity degree* of φ is defined by $\sigma(\varphi) := \frac{|\mathrm{Im}\varphi|}{|H|}$.

This invariant quantifies how much of the codomain is covered by the homomorphism. It satisfies $0 < \sigma(\varphi) \leq 1$, and $\sigma(\varphi) = 1$ if and only if φ is surjective.

These probabilistic invariants capture different structural aspects of finite groups and mappings between them. The commutativity degree reflects internal commutative behavior, the relative commutativity degree measures subgroup-group interaction, the homomorphism degree quantifies approximate multiplicativity of arbitrary maps, and the surjectivity degree measures the extent to which a homomorphism covers its codomain.

Motivated by these quantitative perspectives, especially the homomorphism degree, it is natural to ask how effectively one finite group can be embedded into another by injective maps that preserve the group operation on as many ordered pairs as possible. This leads to the notion of the embeddability degree, which will be introduced and studied in the next section.

3. The Embeddability Degree

This section introduces our new probabilistic invariant for finite groups, called the embeddability degree, and investigates some of its basic properties. Throughout this section, all groups are assumed to be finite. We first recall the homomorphism degree of an arbitrary function between finite groups.

Definition 3.1.: Let G and H be finite groups. The *embeddability degree* of G into H is defined by

$$\eta(G, H) := \max\{\chi(f): f: G \to H \text{ is injective}\}. \tag{3.1}$$

provided that at least one injective function from G to H exists. If no such injective function exists, we define $\eta(G, H) := 0$. Since G and H are finite groups, the set of injective functions $f: G \to H$ is finite. Hence, the supremum is actually attained, and we may use maximum instead of supremum.

The invariant $\eta(G, H)$ measures how closely G can be embedded into H by an injective function which behaves like a homomorphism on as many ordered pairs as possible.

Although the following two propositions are straightforward, we include their proof since they illustrate the method of computing the embeddability degree and clarify the role of the defining condition in a simple but fundamental case.

Proposition 3.2. Let G and H be finite groups. Then, $\eta(G,H) = 1$ if and only if G embeds into H, that is, if and only if there exists an injective group homomorphism from G to H.

Proof. Suppose first that G embeds into H. Then, there exists an injective group homomorphism $f: G \to H$. Since f is a homomorphism, we have $f(ab) = f(a)f(b)$ for all $a, b \in G$. Therefore, every ordered pair $(a, b) \in G^2$ satisfies the homomorphism identity. Hence, $\chi(f) = \frac{|G|^2}{|G|^2} = 1$. Thus, $\eta(G,H) = 1$.

Conversely, suppose that $\eta(G,H) = 1$. Then, there exists an injective function $f: G \to H$ such that $\chi(f) = 1$. This means that

$$|\{(a, b) \in G^2 : f(ab) = f(a)f(b)\}| = |G|^2.$$

Hence every pair $(a, b) \in G^2$ satisfies $f(ab) = f(a)f(b)$. Therefore, f is a group homomorphism. Since f is injective by definition of $\eta(G,H)$, f is an embedding of G into H. □

Proposition 3.3.: Let G and H be finite groups. Then, $\eta(G,H) = 0$ if and only if $|G| > |H|$.

Proof. If $|G| > |H|$, then there is no injective function from G to H. Therefore, by definition, $\eta(G,H) = 0$.

Conversely, suppose that $|G| \leq |H|$. Then, there exists at least one injective function $f: G \to H$. Choose such a function with $f(e_G) = e_H$. Then, for the pair (e_G, e_G), we have

$$f(e_G e_G) = f(e_G) = e_H$$

and

$$f(e_G)f(e_G) = e_H e_H = e_H.$$

Thus, the pair (e_G, e_G) satisfies the homomorphism identity. Hence, $\chi(f) \geq \frac{1}{|G|^2}$. Therefore, $\eta(G,H) \geq \frac{1}{|G|^2} > 0$. So, $\eta(G,H) = 0$ cannot occur when $|G| \leq |H|$. This proves the equivalence. □

Corollary 3.4.: If $|G| \leq |H|$, then $\eta(G,H) \geq \frac{1}{|G|^2}$.

The following proposition shows that when the two groups have the same cardinality, the embeddability degree becomes symmetric, reflecting a natural reciprocity between the two groups.

Proposition 3.5.: Let G and H be finite groups such that $|G| = |H|$. Then, $\eta(G,H) = \eta(H,G)$.

Proof. Since $|G| = |H|$, every injective function $f: G \to H$ is bijective. Let $f: G \to H$ be a bijection. We show that $\chi(f) = \chi(f^{-1})$. Let $(a, b) \in G^2$, and set $x = f(a)$, $y = f(b)$. Then, $(x, y) \in H^2$. Now $f(ab) = f(a)f(b)$ if and only if $f(ab) = xy$.

Applying f^{-1} to both sides gives $ab = f^{-1}(xy)$. Since $a = f^{-1}(x)$ and $b = f^{-1}(y)$, this is equivalent to

$$f^{-1}(xy) = f^{-1}(x)f^{-1}(y).$$

Thus, the pair (a, b) is a homomorphic pair for f if and only if the pair $(x, y) = (f(a), f(b))$ is a homomorphic pair for f^{-1}.

Since the map

$$G^2 \to H^2, \qquad (a, b) \mapsto (f(a), f(b))$$

is a bijection, the number of successful pairs for f equals the number of successful pairs for f^{-1}. Therefore, $\chi(f) = \chi(f^{-1})$. Taking the maximum over all bijections $f: G \to H$, we obtain

$\eta(G, H) = \eta(H, G)$. □

The following theorem establishes a fundamental composition inequality for the embeddability degree, relating the embeddability behavior across three finite groups through the interaction of optimal injective maps.

Theorem 3.6.: Let G, H, K be finite groups. Then,

$$\eta(G, K) \geq \max\left\{0,\ \eta(G, H) + (\eta(H, K) - 1)\frac{|H|^2}{|G|^2}\right\}. \qquad (3.2)$$

Proof. If either $\eta(G, H) = 0$ or $\eta(H, K) = 0$, then the asserted inequality is trivial whenever the right-hand side is non-positive. Hence, we may assume that

$$\eta(G, H) > 0 \qquad \text{and} \qquad \eta(H, K) > 0.$$

Then, there exist injective functions $f: G \to H$, $g: H \to K$ such that

$$\chi(f) = \eta(G, H) \qquad \text{and} \qquad \chi(g) = \eta(H, K).$$

Since f and g are injective, the composition $g \circ f: G \to K$ is also injective.

Define $S_f := \{(a, b) \in G^2 : f(ab) = f(a)f(b)\}$. Then, $|S_f| = \chi(f)|G|^2$. Consider the map

$$G^2 \to H^2, \qquad (a, b) \mapsto (f(a), f(b)).$$

Since f is injective, this map is also injective. Therefore, the set

$$T_f := \{(f(a), f(b)) : (a, b) \in S_f\} \subseteq H^2$$

has cardinality $|T_f| = |S_f| = \chi(f)|G|^2$. Now define

$$S_g := \{(x, y) \in H^2 : g(xy) = g(x)g(y)\}.$$

Then, $|S_g| = \chi(g)|H|^2$. Both T_f and S_g are subsets of H^2. Hence, $|T_f \cap S_g| \geq |T_f| + |S_g| - |H|^2$. Therefore, $|T_f \cap S_g| \geq \chi(f)|G|^2 + \chi(g)|H|^2 - |H|^2$.

Now take $(a,b) \in G^2$ such that $(f(a),f(b)) \in T_f \cap S_g$. Since $(f(a),f(b)) \in T_f$, we have $f(ab) = f(a)f(b)$. Since $(f(a),f(b)) \in S_g$, we have $g(f(a)f(b)) = g(f(a))g(f(b))$.

Combining these two equalities gives

$(g \circ f)(ab) = g(f(ab)) = g(f(a)f(b)) = g(f(a))g(f(b)) = (g \circ f)(a)(g \circ f)(b)$.

Thus, every element of $T_f \cap S_g$ gives a successful homomorphic pair for $g \circ f$.

Consequently,

$$\chi(g \circ f) \geq \frac{\chi(f)|G|^2 + \chi(g)|H|^2 - |H|^2}{|G|^2}.$$

Hence,

$$\chi(g \circ f) \geq \chi(f) + (\chi(g) - 1)\frac{|H|^2}{|G|^2}.$$

Substituting $\chi(f) = \eta(G,H)$, $\chi(g) = \eta(H,K)$, we obtain

$$\chi(g \circ f) \geq \eta(G,H) + (\eta(H,K) - 1)\frac{|H|^2}{|G|^2}.$$

Since $g \circ f$ is injective, we have $\eta(G,K) \geq \chi(g \circ f)$. Thus,

$$\eta(G,K) \geq \eta(G,H) + (\eta(H,K) - 1)\frac{|H|^2}{|G|^2}.$$

If the right-hand side is negative, the trivial bound $\eta(G,K) \geq 0$ gives the stated maximum form. □

4. Explicit computations for small cyclic groups

The following examples illustrate the computation of the embeddability degree for several small cyclic groups. Throughout this subsection, cyclic groups are written additively.

Example 4.1.: We compute $\eta(\mathbb{Z}_2, \mathbb{Z}_3)$. Let $\mathbb{Z}_2 = \{0,1\}$, $\mathbb{Z}_3 = \{0,1,2\}$. There are exactly six injective functions from $\mathbb{Z}_2$ to $\mathbb{Z}_3$. We list them and compute $\chi(f)$ for each.

Let $f(0) = a$, $f(1) = b$, where $a, b \in \mathbb{Z}_3$ and $a \neq b$. The homomorphism condition is $f(x+y) = f(x) + f(y)$. The four ordered pairs in $\mathbb{Z}_2^2$ are $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.

For these pairs, the corresponding conditions are: $(0,0)$: $f(0) = f(0) + f(0)$, that is $a = 2a$. In $\mathbb{Z}_3$, this is equivalent to $a = 0$.

For $(0,1)$, we get $f(1) = f(0) + f(1)$, that is $b = a + b$, again equivalent to $a = 0$.

For (1,0), we get $f(1) = f(1) + f(0)$, that is $b = b + a$, again equivalent to $a = 0$.

For (1,1), since $1 + 1 = 0$ in $\mathbb{Z}_2$, we get $f(0) = f(1) + f(1)$, that is $a = 2b$.

Now we compute all injective functions. Table 1 below summarizes the computation of $\chi(f)$ for all injective functions $f: \mathbb{Z}_2 \to \mathbb{Z}_3$.

Table 1. Homomorphism degrees of all injective functions from $\mathbb{Z}_2$ to $\mathbb{Z}_3$

f(0)	f(1)	successful pairs	χ(f)
0	1	(0,0), (0,1), (1,0)	$\frac{3}{4}$
0	2	(0,0), (0,1), (1,0)	$\frac{3}{4}$
1	0	none	0
1	2	(1,1)	$\frac{1}{4}$
2	0	none	0
2	1	(1,1)	$\frac{1}{4}$

Therefore, the maximum value is $\frac{3}{4}$. Hence $\eta(\mathbb{Z}_2, \mathbb{Z}_3) = \frac{3}{4}$.

Example 4.2.: We compute $\eta(\mathbb{Z}_2, \mathbb{Z}_5)$. Let $\mathbb{Z}_2 = \{0,1\}, \mathbb{Z}_5 = \{0,1,2,3,4\}$. Let $f: \mathbb{Z}_2 \to \mathbb{Z}_5$ be injective and write $f(0) = a,\ f(1) = b$, with $a \neq b$.

As in the previous example, the first three pairs (0,0), (0,1), (1,0) are successful if and only if $a = 0$. The last pair (1,1) is successful if and only if $a = 2b$ in $\mathbb{Z}_5$.

If $a = 0$, then the first three pairs are successful. Since $b \neq 0$ and $2b = 0$ has no nonzero solution in $\mathbb{Z}_5$, the pair (1,1) is not successful. Therefore, $\chi(f) = \frac{3}{4}$ for every injective f with $f(0) = 0$.

If $a \neq 0$, then the first three pairs are not successful. At most the pair (1,1) can be successful. Hence, $\chi(f) \leq \frac{1}{4}$. Thus the maximum value is $\frac{3}{4}$. Therefore $\eta(\mathbb{Z}_2, \mathbb{Z}_5) = \frac{3}{4}$.

Theorem 4.3.: Let $n \geq 2$. Then, $\eta(\mathbb{Z}_2, \mathbb{Z}_n) = \begin{cases} 1, & 2 \mid n, \\ \frac{3}{4}, & 2 \nmid n. \end{cases}$

Proof. Let $f: \mathbb{Z}_2 \to \mathbb{Z}_n$ be injective, and write $f(0) = a,\ f(1) = b$, with $a \neq b$. The four ordered pairs in $\mathbb{Z}_2^2$ yield the following conditions:

$$(0,0): \quad a = 2a,$$

$$(0,1): \quad b = a + b,$$

$$(1,0): \quad b = b + a,$$

$$(1,1): \quad a = 2b.$$

The first three conditions are all equivalent to $a = 0$.

If $2 \mid n$, then $\mathbb{Z}_n$ contains an element of order 2, namely $n/2$. Define $f(0) = 0$, $f(1) = \frac{n}{2}$.

Then, $2f(1) = 2 \cdot \frac{n}{2} = n = 0$ in $\mathbb{Z}_n$. Hence,

$$f(1 + 1) = f(0) = 0 = 2f(1) = f(1) + f(1).$$

The other three conditions are also satisfied because $f(0) = 0$. Therefore, f is an injective homomorphism, and $\eta(\mathbb{Z}_2, \mathbb{Z}_n) = 1$.

Now suppose that $2 \nmid n$. If $a = 0$, then the first three pairs are successful. Since $b \neq 0$, the equation $2b = 0$ has no nonzero solution in $\mathbb{Z}_n$, because 2 is invertible modulo n. Therefore, the pair (1,1) is not successful, and $\chi(f) = \frac{3}{4}$.

If $a \neq 0$, then the first three conditions fail. Only the last condition $a = 2b$ may hold, and therefore $\chi(f) \leq \frac{1}{4}$. Thus, the maximum value is $\frac{3}{4}$. Consequently, $(\mathbb{Z}_2, \mathbb{Z}_n) = \frac{3}{4}$ whenever n is odd. □

Example 4.4.: We compute $\eta(\mathbb{Z}_3, \mathbb{Z}_4)$. Let $\mathbb{Z}_3 = \{0,1,2\}$, $\mathbb{Z}_4 = \{0,1,2,3\}$. Let $f: \mathbb{Z}_3 \to \mathbb{Z}_4$ be injective and write $f(0) = a$, $f(1) = b$, $f(2) = c$, where a, b, c are distinct elements of $\mathbb{Z}_4$.

There are nine ordered pairs in $\mathbb{Z}_3^2$: $(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)$.

We compute the homomorphism condition for each pair.

For (0,0): $f(0 + 0) = f(0) = a$, while $f(0) + f(0) = a + a = 2a$. Thus, (0,0) is successful if and only if $a = 2a$, equivalently $a = 0$.

For (0,1): $f(0 + 1) = f(1) = b$, while $f(0) + f(1) = a + b$. Thus, (0,1) is successful if and only if $b = a + b$, equivalently $a = 0$.

For (1,0), similarly, $f(1 + 0) = f(1) = b$, and $f(1) + f(0) = b + a$. Thus, (1,0) is successful if and only if $a = 0$.

For (0,2) and (2,0), the same argument gives again the condition $a = 0$.

Therefore, the five pairs $(0,0), (0,1), (1,0), (0,2), (2,0)$ are successful if and only if $a = 0$.

Now consider the remaining four pairs.

For $(1,1)$, since $1+1=2$ in $\mathbb{Z}_3$, we get $f(1+1)=f(2)=c$, while $f(1)+f(1)=b+b=2b$. Thus $(1,1)$ is successful if and only if $c=2b$.

For $(1,2)$, since $1+2=0$ in $\mathbb{Z}_3$, we get $f(1+2)=f(0)=a$, while $f(1)+f(2)=b+c$. Thus $(1,2)$ is successful if and only if $a=b+c$.

For $(2,1)$, the same condition appears: $a=c+b$. Since $\mathbb{Z}_4$ is abelian, this is equivalent to $a=b+c$.

For $(2,2)$, since $2+2=1$ in $\mathbb{Z}_3$, we get $f(2+2)=f(1)=b$, while $f(2)+f(2)=c+c=2c$. Thus, $(2,2)$ is successful if and only if $b=2c$.

To obtain many successful pairs, we choose $a=0$. Then, the first five pairs are automatically successful. We need to maximize the number of successful pairs among the remaining four.

Choose $f(0)=0,\ f(1)=1,\ f(2)=3$. Then, $a=0,\ b=1$, and $c=3$. We check the remaining four pairs:

$(1,1)$: $c=3, 2b=2$. So, $(1,1)$ is not successful.

$(1,2)$: $a=0,\ b+c=1+3=4=0$ in $\mathbb{Z}_4$. Hence, $(1,2)$ is successful.

$(2,1)$: $a=0,\ c+b=3+1=4=0$. Hence, $(2,1)$ is successful.

$(2,2)$: $b=1,\ 2c=2\cdot 3=6=2$ in $\mathbb{Z}_4$. Hence, $(2,2)$ is not successful.

Thus, exactly seven pairs are successful: $(0,0),(0,1),(1,0),(0,2),(2,0),(1,2),(2,1)$. Therefore, $\chi(f)=\frac{7}{9}$. So, $\eta(\mathbb{Z}_3,\mathbb{Z}_4)\geq\frac{7}{9}$.

We now show that no injective function can give a larger value. If $a\neq 0$, then the first five pairs fail. Hence, at most four pairs can be successful, so $\chi(f)\leq\frac{4}{9}<\frac{7}{9}$. Thus, a maximizing function must satisfy $a=0$.

Assume $a=0$. Then b,c are distinct nonzero elements of $\mathbb{Z}_4$. The remaining conditions are $c=2b, b+c=0, b=2c$. The middle condition $b+c=0$ gives two successful pairs, namely $(1,2)$ and $(2,1)$. The other two conditions give at most one successful pair each.

We claim that when $a=0$, at most two of the remaining four pairs can be successful. Indeed, if $c=2b$ and $b+c=0$ hold simultaneously, then $b+2b=0$, so $3b=0$. In $\mathbb{Z}_4$, this implies $b=0$, because 3 is invertible modulo 4, contradicting $b\neq 0$. Similarly, the simultaneous validity of $b=2c$ and $b+c=0$ would imply $3c=0$, hence $c=0$, a contradiction. Finally, if $c=2b$ and $b=2c$ hold simultaneously, then $b=2(2b)=4b=0$, again a contradiction because $b\neq 0$.

Thus, among the last four pairs, at most the two middle pairs $(1,2)$ and $(2,1)$ can be simultaneously successful. Therefore, the total number of successful pairs is at most $5+2=7$. Hence, $\eta(\mathbb{Z}_3,\mathbb{Z}_4)=\frac{7}{9}$.

Example 4.5: We compute $\eta(\mathbb{Z}_3, \mathbb{Z}_5)$. Let $\mathbb{Z}_3 = \{0,1,2\}$, $\mathbb{Z}_5 = \{0,1,2,3,4\}$. Let $f(0) = a, f(1) = b, f(2) = c$, with a, b, c distinct.

As in the previous example, the first five pairs $(0,0), (0,1), (1,0), (0,2), (2,0)$ are successful if and only if $a = 0$. The remaining four pairs give the conditions $(1,1)$: $c = 2b$, $(1,2)$: $a = b + c$, $(2,1)$: $a = b + c$, $(2,2)$: $b = 2c$

Choose $f(0) = 0$, $f(1) = 1$, $f(2) = 4$. Then, $a = 0$, $b = 1$, and $c = 4$. The first five pairs are successful.

For the remaining four pairs:

$(1,1)$: $c = 4$, $2b = 2$. So $(1,1)$ is not successful.

$(1,2)$: $a = 0$, $b + c = 1 + 4 = 5 = 0$. So $(1,2)$ is successful.

$(2,1)$: $a = 0$, $c + b = 4 + 1 = 5 = 0$. So $(2,1)$ is successful.

$(2,2)$: $b = 1$, $2c = 2 \cdot 4 = 8 = 3$ in $\mathbb{Z}_5$. Thus $(2,2)$ is not successful.

Hence, exactly seven pairs are successful, and $\chi(f) = \frac{7}{9}$. Therefore, $\eta(\mathbb{Z}_3, \mathbb{Z}_5) \geq \frac{7}{9}$.

We now prove maximality. If $a \neq 0$, then the first five pairs fail, so at most four pairs are successful. Thus, $\chi(f) \leq \frac{4}{9} < \frac{7}{9}$. Hence, any maximizing function must satisfy $a = 0$.

Assume $a = 0$. Then b, c are distinct nonzero elements of $\mathbb{Z}_5$. The remaining conditions are $c = 2b$, $b + c = 0$, $b = 2c$. The condition $b + c = 0$ gives two successful pairs, namely $(1,2)$ and $(2,1)$.

We show that no other remaining condition can hold simultaneously with it. If $c = 2b$ and $b + c = 0$, then $b + 2b = 0$, so $3b = 0$. Since 3 is invertible modulo 5, this gives $b = 0$, contradicting injectivity because $a = 0$ and $b \neq a$.

Similarly, if $b = 2c$ and $b + c = 0$, then $2c + c = 0$, so $3c = 0$, which implies $c = 0$, again impossible.

Finally, if $c = 2b$ and $b = 2c$, then $b = 2(2b) = 4b$, so $3b = 0$. Again $b = 0$, contradiction.

Therefore, at most two of the last four pairs can be successful. Hence, the total number of successful pairs is at most $5 + 2 = 7$. Thus $\eta(\mathbb{Z}_3, \mathbb{Z}_5) = \frac{7}{9}$.

Theorem 4.6.: Let $n \geq 3$. Then $\eta(\mathbb{Z}_3, \mathbb{Z}_n) = \begin{cases} 1, & 3 \mid n, \\ \frac{7}{9}, & 3 \nmid n. \end{cases}$

Proof. Let $f: \mathbb{Z}_3 \to \mathbb{Z}_n$ be injective and write

$$f(0) = a, \qquad f(1) = b, \qquad f(2) = c,$$

where a, b, c are distinct elements of $\mathbb{Z}_n$.

As computed above, the five pairs $(0,0), (0,1), (1,0), (0,2), (2,0)$ are successful if and only if $a = 0$. The remaining four pairs give the conditions

$$(1,1): \quad c = 2b,$$

$$(1,2): \quad a = b + c,$$

$$(2,1): \quad a = b + c,$$

$$(2,2): \quad b = 2c.$$

If $3 \mid n$, then $\mathbb{Z}_n$ contains an element of order 3, namely $n/3$. Define

$$f(0) = 0, \qquad f(1) = \frac{n}{3}, \qquad f(2) = \frac{2n}{3}.$$

Then, f is an injective group homomorphism from $\mathbb{Z}_3$ into $\mathbb{Z}_n$. Hence, $\eta(\mathbb{Z}_3, \mathbb{Z}_n) = 1$.

Now suppose that $3 \nmid n$. We show that $\eta(\mathbb{Z}_3, \mathbb{Z}_n) = \frac{7}{9}$.

First choose any nonzero $b \in \mathbb{Z}_n$ such that $2b \neq 0$, and define $a = 0,\ c = -b$.

For example, one may take $b = 1$ whenever $n \geq 4$. Then,

$$f(0) = 0, \qquad f(1) = b, \qquad f(2) = -b$$

is injective, because $b \neq 0$ and $b \neq -b$. The latter inequality follows from

$$b = -b \Longrightarrow 2b = 0,$$

which would force $b = 0$ when n is odd, and can be avoided by choosing b appropriately when n is even and $3 \nmid n$. With this choice, the first five pairs are successful, and the two pairs $(1,2)$ and $(2,1)$ are also successful because $b + c = b + (-b) = 0 = a$.

Thus, at least seven pairs are successful, so $\eta(\mathbb{Z}_3, \mathbb{Z}_n) \geq \frac{7}{9}$.

We now prove that more than seven successful pairs are impossible. If $a \neq 0$, then the first five pairs fail. Therefore, at most four pairs can be successful, and hence $\chi(f) \leq \frac{4}{9} < \frac{7}{9}$.

Thus, any maximizing function must have $a = 0$.

Assume $a = 0$. Then b, c are distinct nonzero elements of $\mathbb{Z}_n$. The remaining conditions are

$$c = 2b, \qquad b + c = 0, \qquad b = 2c.$$

If both $c = 2b$ and $b + c = 0$ hold, then $b + 2b = 0$, so $3b = 0$. Since $3 \nmid n$, multiplication by 3 is injective on $\mathbb{Z}_n$. Therefore $b = 0$, a contradiction.

Similarly, if both $b = 2c$, and $b + c = 0$ hold, then $3c = 0$, and $c = 0$, a contradiction.

Finally, if both $c = 2b$, and $b = 2c$ hold, then $b = 2c = 2(2b) = 4b$, so $3b = 0$. Again $b = 0$, a contradiction. Therefore, at most one of the single-pair conditions

$$c = 2b, \qquad b = 2c$$

can hold, and neither can hold together with the two-pair condition $b + c = 0$.

The best possibility is the condition $b + c = 0$, which gives exactly two successful pairs among the last four. Hence, the maximum number of successful pairs is $5 + 2 = 7$.

Therefore, $\eta(\mathbb{Z}_3, \mathbb{Z}_n) = \frac{7}{9}$. □

Example 4.7.: We illustrate the composition inequality using $G = \mathbb{Z}_2$, $H = \mathbb{Z}_3$, $K = \mathbb{Z}_5$. From the computations above, $\eta(\mathbb{Z}_2, \mathbb{Z}_3) = \frac{3}{4}$, $\eta(\mathbb{Z}_3, \mathbb{Z}_5) = \frac{7}{9}$, and $\eta(\mathbb{Z}_2, \mathbb{Z}_5) = \frac{3}{4}$. The composition inequality gives $\eta(\mathbb{Z}_2, \mathbb{Z}_5) \geq \eta(\mathbb{Z}_2, \mathbb{Z}_3) + (\eta(\mathbb{Z}_3, \mathbb{Z}_5) - 1)\frac{|\mathbb{Z}_3|^2}{|\mathbb{Z}_2|^2}$. Substituting the values gives $\eta(\mathbb{Z}_2, \mathbb{Z}_5) \geq \frac{3}{4} + \left(\frac{7}{9} - 1\right)\frac{9}{4}$. Since $\frac{7}{9} - 1 = -\frac{2}{9}$, we get $\left(\frac{7}{9} - 1\right)\frac{9}{4} = -\frac{2}{9} \cdot \frac{9}{4} = -\frac{1}{2}$. Therefore, $\eta(\mathbb{Z}_2, \mathbb{Z}_5) \geq \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. Indeed, $\eta(\mathbb{Z}_2, \mathbb{Z}_5) = \frac{3}{4} \geq \frac{1}{4}$. Thus, the composition inequality is valid in this example, although it gives a non-sharp lower bound.

5. Conclusion

In this chapter, we introduced the embeddability degree as a new probabilistic invariant associated with pairs of finite groups. The main motivation was to move beyond the classical all-or-nothing notion of group embedding and to consider a quantitative measure of how closely one finite group can be represented inside another through injective maps that preserve the group operation as frequently as possible.

After establishing the necessary probabilistic background, we developed the basic framework of this invariant and examined several of its first structural properties. In particular, we showed that the embeddability degree recovers the classical notion of embedding in the extremal case, vanishes precisely when injective maps cannot exist for cardinality reasons, and exhibits a natural symmetry when the groups involved have the same order. We also obtained a composition-type inequality that connects the embeddability behavior of three finite groups.

The explicit computations for small cyclic groups show that even in elementary cases, the invariant is capable of reflecting nontrivial algebraic behavior. These examples suggest that the arithmetic structure of the groups involved plays an important role in determining the embeddability degree, and that concrete calculations may reveal broader patterns worthy of further study.

The results presented here should be viewed as a first step rather than a complete theory. Several natural questions remain open. For instance, it would be interesting to determine exact values of the embeddability degree for wider classes of finite groups, to understand how the invariant behaves under standard group-theoretic constructions, and to investigate possible connections with other probabilistic invariants already studied in the literature.

More broadly, the embeddability degree offers a way to examine approximate structural compatibility between finite groups from a quantitative perspective. It is our hope that this notion may provide a useful framework for further work at the intersection of probabilistic group theory and the study of group embeddings.

References

Allahverdiev, B. P., Tuna, H., & Kocabıyık, M. (2026). Sampling Theory Associated with Fractal Sturm–Liouville Equations. *Mathematical Methods in the Applied Sciences*, 1-18.

Arvasi, Z., Çağlayan, E. I., & Odabaş, A. (2022). Commutativity degree of crossed modules. *Turkish Journal of Mathematics*, 46(1), 242–256.

Barzegar, R., Erfanian, A., & Farrokhi, M. D. G. (2013). Finite groups with three relative commutativity degrees. *Bulletin of the Iranian Mathematical Society*, 39(2), 271–280.

Chashiani, A., & Rezaei, R. (2021). On the commutativity degree of a group algebra. *Afrika Matematika*, 32(5), 1137–1145.

Çetin, S., & Gürdal, U. (2024). Crossed modules with action. *Ukrainian Mathematical Journal*, 76(4), 649–668.

Erfanian, A., Rezaei, R., & Lescot, P. (2007). On the relative commutativity degree of a subgroup of a finite group. *Communications in Algebra*, 35(12), 4183–4197.

Gallagher, P. X. (1970). The number of conjugacy classes in a finite group. *Mathematische Zeitschrift*, 118, 175–179.

Ghaneei, M., & Azadi, M. (2021). The nth commutativity degree of semigroups. *Journal of Linear and Topological Algebra*, 10(3), 225–233.

Gürdal, M. (2009a). Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra. *Expositiones Mathematicae*, 27(2), 153–160.

Gürdal, M. (2009b). On the extended eigenvalues and extended eigenvectors of shift operator on the Wiener algebra. *Applied Mathematics Letters*, 22(11), 1727–1729.

Gürdal, U., Oğur, Ö., & Çetin, S. (2025). Unima on [0, 1]. *Fuzzy Sets and Systems*, 516, 109439.

Gustafson, W. H. (1973). What is the probability that two group elements commute? *The American Mathematical Monthly*, 80(9), 1031–1034.

Hashemi, M., & Pirzadeh, M. (2022). A generalization of the nth-commutativity degree in finite groups. *Computational Sciences and Engineering*, 2(1), 33–40.

Lescot, P. (1995). Isoclinism classes and commutativity degrees of finite groups. *Journal of Algebra*, 177(3), 847–869.

Lescot, P. (2001). Central extensions and commutativity degree. *Communications in Algebra*, 29, 4451–4460.

Nath, R. K., & Das, A. K. (2010). On a lower bound of commutativity degree. *Rendiconti del Circolo Matematico di Palermo*, 59, 137–142.

Nath, R. K., & Das, A. K. (2011). On generalized commutativity degree of a finite group. *Rocky Mountain Journal of Mathematics*, 41(6), 1987–2000.

Rezaei, R., & Erfanian, A. (2014). A note on the relative commutativity degree of finite groups. *Asian-European Journal of Mathematics*, 7(1), 1450017.

Rusin, D. (1979). What is the probability that two elements of a finite group commute? *Pacific Journal of Mathematics*, 82(1), 237–247.

Tărnăuceanu, M. (2009). Subgroup commutativity degrees of finite groups. *Journal of Algebra*, 321(9), 2508–2520.

Uc, M. (2025). Quantitative relations between commutativity, surjectivity, and homomorphism degrees in finite groups. *Süleyman Demirel Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, 29(3), 705–715.