Chapter 6

A Szász-Durrmeyer Type Approximation Operator Based On Gauss-Appell Polynomials

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Abstract

This chapter introduces a Szász-Durrmeyer type approximation operator generated by Gauss-Appell polynomials. The construction is motivated by the link between Gauss hypergeometric functions, umbral calculus, and Appell polynomial families. After fixing the auxiliary variable as a parameter, the generating function of the Gauss-Appell polynomials is used to define a positive linear operator on the semi-infinite interval. The basic algebraic structure of the operator is examined through its first moments and central moments. These identities make it possible to establish the main approximation properties in a weighted setting. In particular, a Korovkin-type convergence theorem is obtained, showing that the operators approximate continuous functions uniformly on compact subsets. Quantitative estimates are then derived by means of the usual modulus of continuity, Peetre's K-functional, and the second-order modulus of smoothness. A Voronovskaya type asymptotic formula is also proved, which describes the limiting behaviour of the approximation error and clarifies the role of the first two central moments. Finally, a numerical example supported by graphs and pointwise error tables illustrates the convergence behaviour of the proposed operators. The results show that increasing the main parameter improves the approximation, while the additional parameters provide useful local control over the accuracy.

1. INTRODUCTION

In recent decades, hypergeometric functions have attracted renewed attention due to their deep connections with several areas of mathematics, such as algebraic geometry, representation theory, number theory, combinatorics, and related fields. Among these functions, the Gauss hypergeometric function, denoted by ${}_2\mathcal{H}_1(\alpha,\beta;\gamma;u)$, occupies a central place as a classical special function with important applications in complex analysis, differential equations and mathematical physics. It is well known that the Gauss hypergeometric

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function ${}_2\mathcal{H}_1(\alpha,\beta;\gamma;u)$ is a solution of the hypergeometric differential equation [1]

$$u(u-1)y''(u)+(\gamma-(\alpha+\beta+1)u)y'(u)-\alpha\beta y(u)=0, |u|<1, \quad (1)$$

where α,β and γ may be real or complex parameters with $\gamma \notin \{\dots,-2,-1,0\}$. This differential equation is a member of the Fuchsian class on the complex projective line and possesses three regular singular points, namely $u=0,1$ and ∞.

The Gauss hypergeometric function ${}_2\mathcal{H}_1(\alpha,\beta;\gamma;u)$ is represented by the following power series:

$${}_2\mathcal{H}_1(\alpha,\beta;\gamma;u)=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{u^m}{m!}, \qquad |u|<1. \quad (2)$$

The Pochhammer symbol is defined by

$$\begin{cases}(\beta)_m=\beta(\beta+1)\dots(\beta+m-1) & ;m\geq 1\\(\beta)_0=1 & ;m=0\end{cases}.$$

The umbral approach is a well established symbolic method that provides an effective framework for dealing with a wide variety of problems in science and engineering, especially those related to special functions. By using formal symbolic rules and algebraic type operations, this technique helps reveal underlying structures, reduce the complexity of lengthy computations and offer a clearer interpretation of mathematical relations that may be difficult to handle by classical methods. In this respect, the umbral method is particularly useful in the study of special functions, since it facilitates the construction of new forms, the analysis of their operational properties, and the extension of known families. Therefore, it has become an important tool in several applied areas, including physics, fluid mechanics, and dynamical systems.

In a recent study [2], the hypergeometric function was expressed within the framework of umbral calculus by means of an exponential type representation. Appell polynomials have attracted considerable attention in recent years because of their broad range of applications [3]. Various studies have examined their characterization, generalizations and connections with special functions through algebraic and operational methods [4]. Further research has also investigated noncommutative extensions, generating functions, combinatorial structures and applications in mathematical physics [5,6]. In addition, confluent Appell polynomials and their main properties have been introduced and studied in [7].

An Appell sequence $\{T_m(u)\}_{m\geq 0}$ is commonly defined by means of an exponential generating function of the form [8]:

$$\mathbb{A}(t)e^{ut}=\sum_{k=0}^{\infty}T_k(u)\frac{t^k}{k!}, \quad (3)$$

where the function $\mathbb{A}(t)$ is given by

$$\mathbb{A}(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}, a_0 \neq 0. \tag{4}$$

Using umbral methods, the authors in [9] established a connection between hypergeometric functions and the Appell family. In this way, they derived the generating function for the hypergeometric Appell polynomials in the following form:

$$\sum_{m=0}^{\infty} {}_2\mathcal{H}_1 T_m^{(\alpha,\beta;\gamma)}(u) \frac{t^m}{m!} = \mathbb{A}(t)\ {}_2\mathcal{H}_1(\alpha,\beta;\gamma;ut). \tag{5}$$

They also discussed the main properties of this broad class of special polynomials.

In addition, the authors of [9] extended their study by defining the two variable hypergeometric Appell polynomials through umbral techniques as follows:

$$\sum_{m=0}^{\infty} T_m^{(\alpha,\beta;\gamma)}(u,v) \frac{t^m}{m!} = \mathbb{A}(t) e^{ut}\ {}_2\mathcal{H}_1(\alpha,\beta;\gamma;vt^2), \qquad |ut^2| < 1 \tag{6}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\gamma \notin \{\dots, -2, -1, 0\}$.

In recent years, special functions arising in mathematical physics have been widely developed in both generalized and multivariable settings. In particular, special polynomials involving two variables have provided useful analytical tools for studying various classes of partial differential equations that appear in physical models [10]. Many such special functions and their extensions have been motivated by problems originating from physical phenomena.

One of the main contributions of the present study is the construction of approximation operators based on the Gauss Appell polynomials. These operators are proved to approximate functions defined on a semi infinite interval in a suitable weighted function space. Their convergence behavior is further illustrated by a numerical example and the study ends with concluding remarks.

2. MAIN RESULTS

In this section, we construct a family of positive linear Szász-Durrmeyer type operators generated by the Gauss-Appell polynomials. To this end, we fix the auxiliary variable v as a parameter h. Then the generating function of the Gauss Appell polynomials $T_m^{(\alpha,\beta;\gamma)}(u,h)$ is given by

$$\sum_{k=0}^{\infty} T_k^{(\alpha,\beta;\gamma)}(u,h) \frac{t^k}{k!} = \mathbb{A}(t) e^{ut}\ {}_2\mathcal{H}_1(\alpha,\beta;\gamma;ht^2), \qquad |ht^2| < 1 \tag{7}$$

Since the coefficients of the generating function are non-negative under the assumptions and $\gamma > \beta > \alpha > 0, 0 < h < 1, T_k^{(\alpha,\beta;\gamma)}(mu,h) > 0$ and $\mathbb{A}(t)$ has non negative coefficients with $\mathbb{A}(1) > 0$, the operator defined below is positive and linear:

$$\wp_m(g;u) = \frac{me^{-mu}}{\mathbb{A}(1)\ {}_2\mathcal{H}_1(\alpha,\beta;\gamma;h)} \sum_{k=0}^{\infty} \frac{T_k^{(\alpha,\beta;\gamma)}(mu,h)}{k!} \int_0^{\infty} e^{-mt} \frac{(mt)^k}{k!} g(t)dt. \quad (8)$$

First, we evaluate several moments and central moments of the family of operators defined in (8).

Lemma 2.1. The first few moments of the operators $\wp_m$ are given as follows:

$$\wp_m\,(1;u) = 1,$$

$$\wp_m\,(t;u) = u + \frac{\Delta_1 + 1}{m},$$

$$\wp_m\,(t^2;u) = u^2 + \frac{(2\Delta_1 + 4)u}{m} + \frac{\Delta_2 + 4\Delta_1 + 2}{m^2}$$

for any $u \in [0,\infty)$.

Here,

$$\mathcal{H}_j := {}_2\mathcal{H}_1(\alpha + j, \beta + j; \gamma + j; h), j = 0,1,2,$$

$$\Delta_1 := \frac{\mathbb{A}'(1)}{\mathbb{A}(1)} + 2h\frac{(\alpha)_1(\beta)_1}{(\gamma)_1}\frac{\mathcal{H}_1}{\mathcal{H}_0},$$

$$\Delta_2 := \frac{\mathbb{A}''(1)}{\mathbb{A}(1)} + 2h\frac{(\alpha)_1(\beta)_1}{(\gamma)_1}\frac{\mathcal{H}_1}{\mathcal{H}_0} + 4h^2\frac{(\alpha)_2(\beta)_2}{(\gamma)_2}\frac{\mathcal{H}_2}{\mathcal{H}_0} + 4h\frac{(\alpha)_1(\beta)_1}{(\gamma)_1}\frac{\mathbb{A}'(1)}{\mathbb{A}(1)}\frac{\mathcal{H}_1}{\mathcal{H}_0}.$$

Proof.

Differentiating the generating function in (7) with respect to t once and twice, respectively, we obtain the following identities;

$$\sum_{k=0}^{\infty} \frac{T_k^{(\alpha,\beta;\gamma)}(mu)}{k!} = \mathbb{A}(t)e^{ut}\ {}_2\mathcal{H}_1(\alpha,\beta;\gamma;ht^2),$$

$$\sum_{k=0}^{\infty} \frac{T_{k+1}^{(\alpha,\beta;\gamma)}(mu)}{k!} = e^{ut}\big(\mathbb{A}'(t) + u\mathbb{A}(t)\big)\ {}_2\mathcal{H}_1(\alpha,\beta;\gamma;ht^2)$$

$$+e^{ut}\frac{2ht(\alpha)_1(\beta)_1}{(\gamma)_1}\mathbb{A}(t)\ {}_2\mathcal{H}_1(\alpha+1,\beta+1;\gamma+1;ht^2)$$

$$\sum_{k=0}^{\infty}\frac{T_{k+2}^{(\alpha,\beta;\gamma)}(mu)}{k!}=e^{ut}\left(\mathbb{A}''(t)+2u\mathbb{A}'(t)+u^2\mathbb{A}(t)\right)\,{}_2\mathcal{H}_1(\alpha,\beta;\gamma;ht^2)$$

$$+e^{ut}\left(\frac{4ht(\alpha)_1(\beta)_1}{(\gamma)_1}\left(\mathbb{A}'(t)+u\mathbb{A}(t)\right)+\frac{2h(\alpha)_1(\beta)_1}{(\gamma)_1}\mathbb{A}(t)\right)\,{}_2\mathcal{H}_1(\alpha+1,\beta+1;\gamma+1;ht^2)$$

$$+e^{ut}\frac{4h^2t^2(\alpha)_2(\beta)_2}{(\gamma)_2}\mathbb{A}(t)\,{}_2\mathcal{H}_1(\alpha+2,\beta+2;\gamma+2;ht^2).$$

Putting $t=1$ and replacing u by mu in the above identities, and then applying the operator defined in (8), we obtain the required moment identities.

Lemma 2.2. For the operators $\wp_m$, the following central moments hold:

$$\wp_m\,(t-u;u)=\frac{\Delta_1+1}{m},$$

$$\wp_m\,((t-u)^2;u)=\frac{2u}{m}+\frac{\Delta_2+4\Delta_1+2}{m^2}.$$

Proof. By using Lemma 2.1, the desired result is obtained.

Remark 2.3. The results are as follows

$$\lim_{m\to\infty} m\wp_m(t-u;u)=\Delta_1+1,$$

$$\lim_{m\to\infty} m\wp_m((t-u)^2;u)=2u$$

for $u>0$. Moreover, for the fourth central moment, we have

$$\wp_m((t-u)^2;u)=O\left(\frac{1}{m}\right)$$

or equivalently,

$$m^2\wp_m((t-u)^4;u)=O(1).$$

Theorem 2.4. For every $g\in H$, we have

$$\lim_{m\to\infty}\wp_m(g;u)=g(u),$$

and the convergence is uniform on each compact subset of $[0,\infty)$ and

$$H=\left\{g\colon g\in C[0,\infty),\ \frac{g(u)}{1+u^2}\text{ has a finite limit as } u\longrightarrow\infty\right\}.$$

Proof. From Lemma 2.1, it follows directly that the operators $\wp_m$ reproduce the test functions 1, t and t^2 asymptotically; that is,

$$\lim_{m\to\infty}\wp_m\left(t^j,u\right)=u^j, j=0,1,2, \tag{9}$$

and this convergence is uniform on each compact subset of [0,∞). Hence, by Korovkin-type theorem of Altomare and Campiti [11], the asserted convergence result is obtained. This completes the proof.

Theorem 2.5. Let $g \in C_B[0,\infty)$ be uniformly continuous on [0,∞). Then, for every $u \in [0,\infty)$, the following estimate holds:

$$|\wp_m(g;u) - g(u)| \le 2\omega\left(g;\sqrt{\wp_m((t-u)^2;u)}\right)$$

where $\omega(g;\cdot)$ denotes the usual modulus of continuity of g.

Proof.

$$|\wp_m(g;u) - g(u)| \le \frac{me^{-mu}}{\mathbb{A}(1)\mathcal{H}_0}\sum_{k=0}^{\infty}\frac{T_k^{(\alpha,\beta;\gamma)}(mu,h)}{k!}\int_0^\infty e^{-mt}\frac{(mt)^k}{k!}|g(t) - g(u)|dt$$

$$\le \frac{me^{-mu}}{\mathbb{A}(1)\mathcal{H}_0}\sum_{k=0}^{\infty}\frac{T_k^{(\alpha,\beta;\gamma)}(mu,h)}{k!}\int_0^\infty e^{-mt}\frac{(mt)^k}{k!}\left(1+\frac{|t-u|}{\delta}\right)\omega(g;\delta)dt$$

$$\le \left\{1+\frac{me^{-mu}}{\mathbb{A}(1)\mathcal{H}_0}\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{T_k^{(\alpha,\beta;\gamma)}(mu,h)}{k!}\int_0^\infty e^{-mt}\frac{(mt)^k}{k!}|t-u|dt\right\}\omega(g;\delta).$$

Estimating the integral term by the Cauchy Schwarz inequality, we obtain

$$|\wp_m(g;u) - g(u)| \le \left\{\left(1+\frac{me^{-mu}}{\mathbb{A}(1)\mathcal{H}_0}\right)\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{T_k^{(\alpha,\beta;\gamma)}(mu,h)}{k!}\left(\int_0^\infty e^{-mt}\frac{(mt)^k}{k!}dt\right)^{\frac{1}{2}}\right.$$

$$\left.\times\left(\int_0^\infty e^{-mt}\frac{(mt)^k}{k!}(t-u)^2dt\right)^{\frac{1}{2}}\right\}\omega(g;\delta).$$

Applying the Cauchy Schwarz inequality once more to the summation gives

$$|\wp_m(g;u) - g(u)| \le \left\{1+\frac{1}{\delta}\left(\frac{me^{-mu}}{\mathbb{A}(1)\mathcal{H}_0}\sum_{k=0}^{\infty}\frac{T_k^{(\alpha,\beta;\gamma)}(mu,h)}{k!}\int_0^\infty e^{-mt}\frac{(mt)^k}{k!}dt\right)^{\frac{1}{2}}\right.$$

$$\left.\left(\frac{me^{-mu}}{\mathbb{A}(1)\mathcal{H}_0}\sum_{k=0}^{\infty}\frac{T_k^{(\alpha,\beta;\gamma)}(mu,h)}{k!}\int_0^\infty e^{-mt}\frac{(mt)^k}{k!}(t-u)^2dt\right)^{\frac{1}{2}}\right\}\times\omega(g;\delta)$$

$$=\left\{1+\frac{1}{\delta}\left(\wp_m(t;u)\right)^{\frac{1}{2}}\left(\wp_m((t-u)^2;u)\right)^{\frac{1}{2}}\right\}\omega(g;\delta)$$

$$=\left\{1+\frac{1}{\delta}\left(\wp_m((t-u)^2;u)\right)^{\frac{1}{2}}\right\}\omega(g;\delta)$$

If we choose $\delta = \sqrt{\wp_m((t-u)^2;u)}$, this completes the proof.

Lemma 2.6. For each $u \in [0,\infty)$ and $g \in C_B[0,\infty)$, we have

$$|\wp_m(g;u)| \le \|g\|.$$

Proof. Let $\|g\| := \sup_{t\ge 0}|g(t)|$. From the definition of $\wp_m$, we obtain

$$\begin{aligned}|\wp_m(g;u)| &\le \left|\frac{me^{-mu}}{\mathbb{A}(1)\mathcal{H}_0}\sum_{k=0}^{\infty}\frac{T_k^{(\alpha,\beta;\gamma)}(mu,h)}{k!}\int_0^\infty e^{-mt}\frac{(mt)^k}{k!}g(t)dt\right| \\ &\le \frac{me^{-mu}}{\mathbb{A}(1)\mathcal{H}_0}\sum_{k=0}^{\infty}\frac{T_k^{(\alpha,\beta;\gamma)}(mu,h)}{k!}\int_0^\infty e^{-mt}\frac{(mt)^k}{k!}|g(t)|dt \\ &\le \|g\|\frac{me^{-mu}}{\mathbb{A}(1)\mathcal{H}_0}\sum_{k=0}^{\infty}\frac{T_k^{(\alpha,\beta;\gamma)}(mu,h)}{k!}\int_0^\infty e^{-mt}\frac{(mt)^k}{k!}dt\end{aligned}$$

Then, $|\wp_m(g;u)| \le \|g\|\wp_m(1;u) = \|g\|$.

Now let $C_B[0,\infty)$ be the space of all bounded continuous functions on $[0,\infty)$, equipped with the supremum norm

$$\|g\|_{C_B[0,\infty)} := \sup_{u\in[0,\infty)}|g(u)|, \quad g \in C_B[0,\infty).$$

For $\mu > 0$, Peetre's K-functional is given by

$$\mathcal{K}(g;\mu) := \inf_{\tau\in C_B^2[0,\infty)}\left\{\|g-\tau\|_{C_B[0,\infty)} + \mu\|\tau\|_{C_B^2[0,\infty)}\right\}.$$

Here, $C_B^2[0,\infty)$ denotes the class of all functions $\tau \in C_B[0,\infty)$ whose first and second derivatives also belong to $C_B[0,\infty)$ and it is equipped with the norm

$$\|\tau\|_{C_B^2[0,\infty)} := \|\tau\|_{C_B[0,\infty)} + \|\tau'\|_{C_B[0,\infty)} + \|\tau''\|_{C_B[0,\infty)}$$

[12].

Theorem 2.7. Let $g \in C_B[0,\infty)$. Then, for every $m \in \mathbb{N}$ and $u \in [0,\infty)$, we have

$$|\wp_m(g;u) - g(u)| \le 2\,\mathcal{K}(g;\mu_m(u)),$$

where $\mathcal{K}(g;\cdot)$ denotes Peetre's K-functional and

$$\mu_m(u) = |\wp_m(t-u;u)| + \frac{1}{2}\wp_m((t-u)^2;u).$$

Proof. Let $\tau \in C_B^2[0,\infty)$ be fixed. For $t \ge 0$, Taylor's formula yields

$$\tau(t) = \tau(u) + (t-u)\tau'(u) + \int_u^t (t-s)\,\tau''(s)ds\,. \tag{10}$$

Applying the operator $\wp_m(\cdot\,;u)$ to (10), we obtain

$$
\begin{aligned}
|\wp_m(\tau;u)-\tau(u)| &\le \left|\wp_m\big((t-u)\tau'(u);u\big)+\wp_m\left(\int_u^t (t-s)\,\tau''(s)ds;u\right)\right| \\
&\le |\tau'(u)||\wp_m(t-u;u)|+\wp_m\left(\left|\int_u^t (t-s)\,\tau''(s)ds\right|;u\right) \\
&\le \|\tau'\|_{C_B[0,\infty)}|\wp_m(t-u;u)| \\
&+\|\tau''\|_{C_B[0,\infty)}\wp_m\left(\int_u^t |t-s|\,ds;u\right) \\
&\le \|\tau'\|_{C_B[0,\infty)}|\wp_m(t-u;u)| \\
&+\frac{1}{2}\|\tau''\|_{C_B[0,\infty)}\wp_m((t-u)^2;u).
\end{aligned}
$$

Consequently,

$$|\wp_m(\tau;u)-\tau(u)| \le \mu_m(u)\|\tau\|_{C_B^2[0,\infty)}\,.$$

Using the preceding estimate together with Lemma 2.6, for any $h \in C_B^2[0,\infty)$ we obtain

$$
\begin{aligned}
&|\wp_m(g;u)-g(u)| \\
&\quad= |\wp_m(g;u)-g(u)+\wp_m(\tau;u)-\wp_m(\tau;u)+\tau(u) \\
&\quad-\tau(u)| \\
&\quad\le |\wp_m(g-\tau;u)|+|g(u)-\tau(u)|+|\wp_m(\tau;u)-\tau(u)| \\
&\quad\le \|g-\tau\|_{C_B[0,\infty)}\wp_m(t;u)+\|g-\tau\|_{C_B[0,\infty)} \\
&\quad+\mu_m(u)\|\tau\|_{C_B^2[0,\infty)} \\
&\quad\le 2\left(\|g-\tau\|_{C_B[0,\infty)}+\mu_m(u)\|\tau\|_{C_B^2[0,\infty)}\right).
\end{aligned}
$$

Taking the infimum over all $\tau \in C_B^2[0,\infty)$ yields

$$|\,\wp_m(g;u)-g(u)| \le 2\,\mathcal{K}\big(g;\mu_m(u)\big). \tag{11}$$

Hence the proof is complete.

For a function $g \in C_B[0,\infty)$, the second order modulus of continuity is defined by

$$\omega_2(g;\delta) := \sup_{0<t\le\delta}\|g(\,\cdot+2t)-2g(\,\cdot+t)+g(\cdot)\|_{C_B[0,\infty)}\,.$$

Moreover, Peetre's K-functional and ω_2 are connected through the estimate

$$\mathcal{K}(g;\delta) \le N\big(\omega_2\big(g;\sqrt{\delta}\big)+min\{1,\delta\}\|g\|_{C_B[0,\infty)}\big), \tag{12}$$

where $N>0$ is a constant independent of f and δ. Combining (11) and (12), we obtain

$$|\wp_m(g;u)-g(u)| \le\; N\left(\omega_2\left(g;\sqrt{\mu_n(u)}\right)+min\{1,\mu_m(u)\}\|g\|_{C_B[0,\infty)}\right).$$

Theorem 2.8. Let $g, g', g'' \in H$. Then, for every $u>0$,

$$\lim_{m\to\infty} m(\wp_m(g;u) - g(u)) = (\Delta_1 + 1)g'(u) + u \cdot g''(u).$$

Moreover, the convergence is uniform on each compact subset of $[0,\infty)$.

Proof. By Taylor's formula, expanding g at the point u, we

$$g(t) = g(u) + g'(u)(t-u) + \frac{g''(u)}{2}(t-u)^2 + (t-u)^2\zeta(t,u), \quad (13)$$

where $\chi(t,u) \to 0$ as $t \to u$. Applying the operator $\wp_m$ to both sides of (13), we obtain

$$\wp_m(g;u) = g(u) + g'(u)\wp_m(t-u;u) + \frac{g''(u)}{2}\wp_m((t-u)^2;u) + \wp_m(\chi(t,u)(t-u)^2;u).$$

Consequently,

$$m\big(\wp_m(g;u) - g(u)\big) = mg'(u)\wp_m(t-u;u) + m\frac{g''(u)}{2}\wp_m((t-u)^2;u) + m\wp_m(\chi(t,u)(t-u)^2;u).$$

It remains to estimate the remainder term. By the Cauchy Schwarz inequality, we have

$$m\big(\wp_m(\chi(t,u)(t-u)^2;u)\big) \le \sqrt{\wp_m(\chi^2(t,u);u)}\sqrt{m^2\wp_m((t-u)^4;u)}.$$

Since $\chi(t,u) \to 0$ as $t \to u$ and by the convergence property of the operators $\wp_m$, we get

$$\lim_{m\to\infty} \wp_m(\chi^2(t,u);u) = \zeta^2(u,u) = 0. \quad (14)$$

Moreover,

$$m^2\wp_m((t-u)^4;u) = O(1),$$

as $m \to \infty$. Hence,

$$\lim_{m\to\infty} m^2\wp_m((t-u)^4;u) = 0.$$

Combining this observation with Remark 2.3 and (14) completes the proof.

3. ILLUSTRATIVE EXAMPLE

To illustrate the approximation behaviour of the proposed Gauss-Appell Szász-Durrmeyer type operators, we consider the test function $g(u) = e^{-u}\sin(3u) + 2,\ \ x \in [0,2]$. This function is continuous, bounded and oscillatory on the considered interval. Therefore, it is suitable for testing the approximation performance of the operator. In the numerical computations, the auxiliary function is chosen as $\mathbb{A}(t) = e^t$. With this choice, the corresponding

Gauss-Appell polynomials are generated consistently by the generating function used in the construction of the operator.

1- The first graph shows the approximation behaviour of the operator for $m = 5, 20, 50, 100$. It is clearly observed that the approximation improves as m increases. For $m = 5$, the approximation curve is relatively far from the original function, especially near the peak region of the function. However, as nnn becomes larger, the operator follows the oscillatory shape of the function more accurately. The error table supports this visual observation. For example, at $u = 0$, the error decreases from 3.74785×10^{-1} for $m = 5$ to 6.1732×10^{-2} for $m = 100$. These results confirm that the proposed operator has a clear convergence tendency with respect to increasing n. The best numerical performance among the tested values is obtained for $m = 100$.

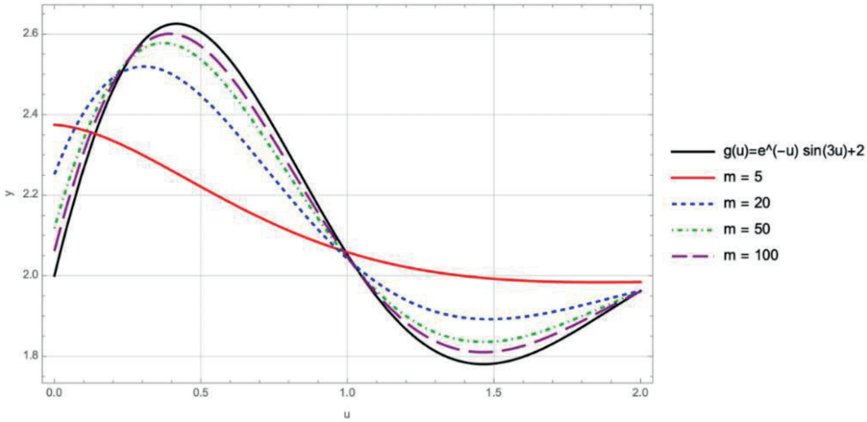

Figure 1. Approximation of $g(u) = e^{-u}\sin(3u) + 2$ by the Gauss-Appell Szász-Durrmeyer type operator for different values of m.

Table 1. Pointwise absolute errors for different values of m.

u	m = 5	m = 20	m = 50	m = 100
0.00	3.74785×10^{-1}	2.51578×10^{-1}	1.18015×10^{-1}	6.1732×10^{-2}
0.25	2.12562×10^{-1}	1.89648×10^{-2}	2.79981×10^{-3}	3.61542×10^{-3}
0.50	3.84127×10^{-1}	1.57665×10^{-1}	6.85793×10^{-2}	3.50561×10^{-2}
0.75	2.39553×10^{-1}	1.26539×10^{-1}	6.19262×10^{-2}	3.3208×10^{-2}
1.00	6.39449×10^{-3}	1.01739×10^{-2}	8.61782×10^{-3}	5.43876×10^{-3}
1.25	1.78437×10^{-1}	8.70334×10^{-2}	4.06492×10^{-2}	2.13641×10^{-2}
1.50	2.10476×10^{-1}	1.1028×10^{-1}	5.46772×10^{-2}	2.95573×10^{-2}
1.75	1.33668×10^{-1}	6.72024×10^{-2}	3.39208×10^{-2}	1.85634×10^{-2}
2.00	2.23362×10^{-2}	7.63995×10^{-4}	4.86117×10^{-4}	1.14479×10^{-3}

2- In the second graph, the influence of the parameter α is investigated. The considered values are $\alpha = 0.20, 1.50, 2.90$. The numerical curves show that changing α has a visible but moderate effect on the approximation. Around the maximum point of the function, the approximation curves remain below the exact function. However, the general shape of the function is preserved for all selected values of α.

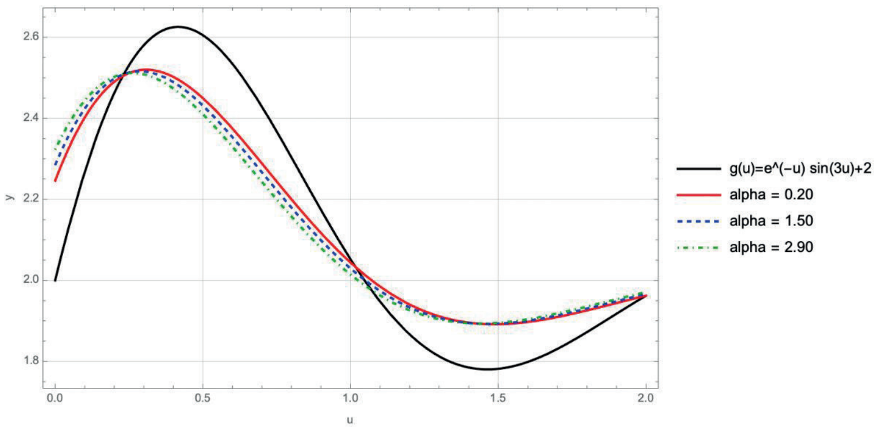

Figure 2. Approximation of $g(u) = e^{-u}\sin(3u) + 2$ for different values of α, while the remaining parameters are fixed.

The error table indicates that the smallest error is not always obtained for the same value of α. For instance, at $u = 0.5$, the errors are $1.55203 \times 10^{-1}, 1.73565 \times 10^{-1}, 1.94723 \times 10^{-1}$, corresponding to $\alpha = 0.20, 1.50, 2.90.$, respectively. Hence, at this point, the smallest error occurs for $\alpha = 0.20$. On the other hand, at $u = 1.25$, the smallest error is obtained for $\alpha = 2.90$. This shows that the parameter α affects the local behaviour of the operator. Therefore, suitable choices of α may improve the approximation quality on particular subintervals.

Table 2. Pointwise absolute errors for different values of α.

u	alpha = 0.20	alpha = 1.50	alpha = 2.90
0.00	2.4596×10^{-1}	2.84682×10^{-1}	3.2242×10^{-1}
0.25	1.93396×10^{-2}	1.79908×10^{-2}	1.95739×10^{-2}
0.50	1.55203×10^{-1}	1.73565×10^{-1}	1.94723×10^{-1}
0.75	1.23574×10^{-1}	1.44946×10^{-1}	1.67962×10^{-1}
1.00	8.09814×10^{-3}	2.27407×10^{-2}	3.7799×10^{-2}
1.25	8.78486×10^{-2}	8.23306×10^{-2}	7.71884×10^{-2}
1.50	1.10132×10^{-1}	1.11435×10^{-1}	1.13375×10^{-1}
1.75	6.66175×10^{-2}	7.09183×10^{-2}	7.57432×10^{-2}
2.00	1.81144×10^{-4}	4.35568×10^{-3}	8.7936×10^{-3}

3- The third graph presents the effect of the parameter β. The selected values are $\beta = 0.90, 1.30, 2.80$. The approximation curves are very close to each other. This indicates that, for the chosen parameter set, the effect of β on the graphical approximation is weaker than the effect of n. Although the curves are not exactly identical, their differences are relatively small.

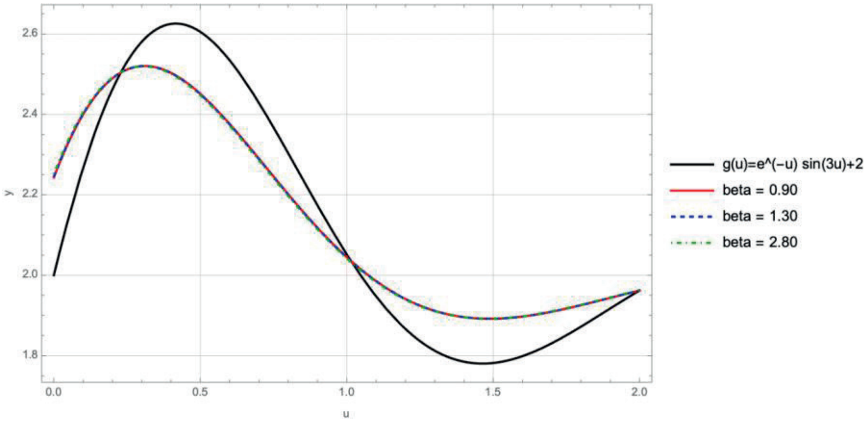

Figure 3. Approximation of $g(u) = e^{-u}\sin(3u) + 2$ for different values of β, while the remaining parameters are fixed.

This conclusion is also supported by the error table. For example, at $u = 1.5$, the errors are $1.10051 \times 10^{-1}, 1.1009 \times 10^{-1}, 1.10259 \times 10^{-1}$. These values are almost the same. Similarly, at $u = 2.0$, the errors remain very small for all tested β values. Therefore, the parameter β has only a limited numerical influence in this example. Nevertheless, small variations are still visible in the table. This means that β can fine tune the approximation, but it does not dominate the convergence behaviour of the operator.

Table 3. Pointwise absolute errors for different values of β.

u	beta = 0.90	beta = 1.30	beta = 2.80
0.00	2.43215×10^{-1}	2.44821×10^{-1}	2.51089×10^{-1}
0.25	1.94855×10^{-2}	1.93606×10^{-2}	1.89635×10^{-2}
0.50	1.53958×10^{-1}	1.54642×10^{-1}	1.57413×10^{-1}
0.75	1.22097×10^{-1}	1.22931×10^{-1}	1.26256×10^{-1}
1.00	7.07402×10^{-3}	7.66245×10^{-3}	9.98417×10^{-3}
1.25	8.82438×10^{-2}	8.80093×10^{-2}	8.71016×10^{-2}
1.50	1.10051×10^{-1}	1.1009×10^{-1}	1.10259×10^{-1}
1.75	6.63237×10^{-2}	6.6487×10^{-2}	6.71441×10^{-2}
2.00	1.08314×10^{-4}	5.60007×10^{-5}	7.08979×10^{-4}

4- The fourth graph investigates the role of the parameter γ. The selected values are $\gamma = 2.10, 3.80, 4.50$. As in the case of the parameter β, the approximation curves corresponding to different γ values are quite close to each other. This suggests that the parameter γ has a relatively mild effect on the approximation for the present test function and interval. The error table confirms this behaviour. At $u = 0.5$, the errors are $1.58789 \times 10^{-1}, 1.55728 \times 10^{-1}, 1.5176 \times 10^{-1}$. Thus, increasing γ slightly improves the approximation at this point. At $u = 2.0$, the error decreases from 1.02214×10^{-3} for $\gamma = 2.10$ to 1.83198×10^{-4} for $\gamma = 4.50$. This shows that larger values of γ may provide better local accuracy near the right endpoint of the interval. Overall, the parameter γ has a stabilizing effect on the approximation, although its influence is not as strong as the influence of m.

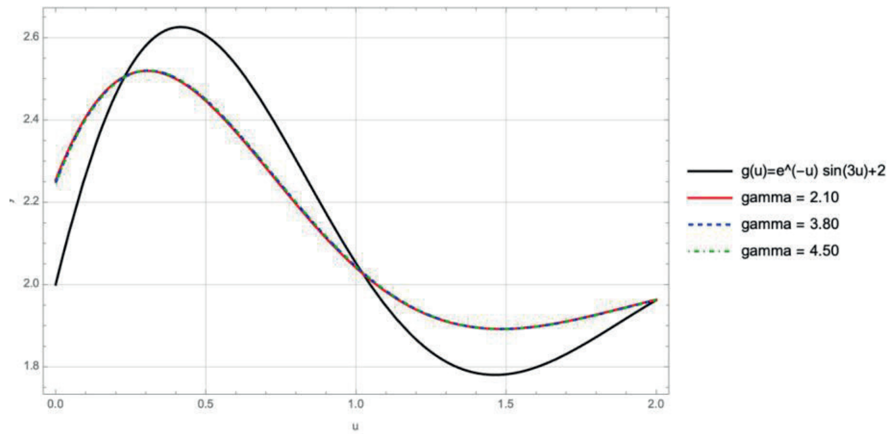

Figure 4. Approximation of $g(u) = e^{-u}\sin(3u) + 2$ for different values of γ, while the remaining parameters are fixed.

Table 4. Pointwise absolute errors for different values of γ.

u	gamma = 2.10	gamma = 3.80	gamma = 4.50
0.00	2.53998×10^{-1}	2.47303×10^{-1}	2.46054×10^{-1}
0.25	1.88593×10^{-2}	1.91937×10^{-2}	1.92729×10^{-2}
0.50	1.58789×10^{-1}	1.55728×10^{-1}	1.55176×10^{-1}
0.75	1.27859×10^{-1}	1.2424×10^{-1}	1.23578×10^{-1}
1.00	1.10831×10^{-2}	8.57915×10^{-3}	8.11646×10^{-3}
1.25	8.66869×10^{-2}	8.76491×10^{-2}	8.783×10^{-2}
1.50	1.10356×10^{-1}	1.10155×10^{-1}	1.10121×10^{-1}
1.75	6.74665×10^{-2}	6.6745×10^{-2}	6.66141×10^{-2}
2.00	1.02214×10^{-3}	3.13327×10^{-4}	1.83198×10^{-4}

4- CONCLUSION

In this chapter, a new Szász-Durrmeyer type approximation operator based on Gauss-Appell polynomials has been constructed and studied. The proposed operator combines the structural properties of Gauss hypergeometric functions with the flexibility of Appell-type polynomial families. This connection provides a useful framework for defining positive linear operators on the semi infinite interval. The fundamental moments and central moments of the operator were obtained explicitly. These results played a central role in the analysis, since they allowed us to investigate the approximation behaviour of the operator in a systematic way. By using these moment identities, a Korovkin type convergence theorem was established. Thus, it was shown that the proposed operators converge to the considered function uniformly on compact subsets of the interval. In addition to the qualitative convergence result, quantitative estimates were obtained by means of the usual modulus of continuity, Peetre's K-functional, and the second order modulus of smoothness. These estimates describe how the approximation error depends on the smoothness of the function and on the central moments of the operator. Furthermore, a Voronovskaya type theorem was proved, giving a more precise asymptotic description of the approximation error. The illustrative example confirmed the theoretical results. The graphs and pointwise error tables showed that the approximation becomes better as the parameter m increases. The numerical results also indicated that the parameters α, β and γ influence the local behaviour of the operator. Among them, some parameters have a stronger effect on the accuracy, while others mainly provide small corrections or stabilization. Overall, the results show that the Gauss-Appell based Szász-Durrmeyer type operators form a meaningful and effective class of positive linear approximation operators. The construction may also be extended in future studies to different polynomial families, weighted spaces, or higher dimensional settings.

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