Combining Principal Normal Indicatrix Curves And Direction Curves With An Alternative Frame

İlkay Arslan Güven[1]

Fatma Çolak[2]

Abstract

This work examined the direction curves of principal normal indicatrix curves regarding a regular curve in Euclidean space of three dimension. The {N,C,W} frame was assigned as a new alternative frame for the mentioned curve. Investigating relationships among the main curve and direction curves using such novel alternative frame is an innovative approach. The Frenet members, curvature, torsion, harmonic curvature and geodesic curvature of direction curves were determined. Direction curve characterizations in the forms of the C-slant helix , slant helix and general helix were provided. In the end, the relevant figures for these curves were displayed.

1. Introduction

The primary field of differential geometry study that has received the most attention is curve theory. Curves come in many forms. Helix curves, such as those found in animal horns, seedpods, and plant shoots, are prevalent in nature. Additionally, the DNA molecule, which makes up the majority of genetic material, is made up of two chains that helix. These chains have couple helix shapes. Further, Salmonella and Escherichia coli flagella have helices. A screw is the most common example of a helix. It has a helical construction that converts rotation to motion along the axis. Computer graphics and the manufacturing of artificial helices from various materials both use helices.

1 Prof. Dr., Gaziantep University Art and Science Faculty Deparment of Mathematics, iarslan@gantep.edu.tr, 0000-0002-5302-6074

2 Master Student, Gaziantep University Graduate School of Natural and Applied Sciences Deparment of Mathematics, fatmacolak33@gmail.com, 0000-0003-4919-4793

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Ellipstic curves are additionally employed in cryptography. The role that curves play in surfaces is their most significant feature. Curves are useful for generating a surface concept, that works in physics, engineering, industry, and other fields.

One of curve theory's most alluring topics is the associated curve. Associated curves are defined as two or more curves that have a mathematical relationship to one another. These curves include Bertrand curve mates, Mannheim curve mates, adjoint curves and involute-evolute curve mates, among others.

Choi and Kim [3] recently suggested adding a new form of associated curve to the literature. They termed it the direction curve and described it such that the integral curve of an each Frenet vector in the mentioned curve. They gave relationships between related curves' curvature and torsion. They provided a canonical method to build these helices and characterized slant and general helices with regards to associated curves. In Minkowski space, direction curves were studied by Qian and Kim in [11]. Inspired by these direction curves, Macit and Düldül [10] utilized a Frenet curve's Darboux vector of a Frenet curve to find W-direction curves. The V-direction curve of a curve that is located on a surface covered by the Darboux frame was also defined. Kızıltuğ and Önder [7] explored the theory of direction curves in compact Lie groups of three dimension. Körpınar et al. [8] used a curve's Bishop frame to provide direction curves over again.

Another type of associated curve that has been extensively studied for a long time are spherical indicatrix curves. Kula and Yaylı [9] researched slant helices' spherical indicatrix curves. They discovered that these spherical helices are also spherical indicatrices. Şahiner [13] investigated the tangent indicatrix's direction curve and the relationship between spherical and direction curves. These curves belong to the allied curves class as well. Specific relationships between the curves' curvatures were provided. Techniques for obtaining the spherical helix, spherical slant helix, and general helix from the circle were discovered. In Şahiner [14], similar scenarios were examined for principal normal indicatrix curves.

Beside this, frames of curves are crucial to look at when studying curve theory because of their specifications. Examples of adapted curve frames abound. The most well-recognized and commonly used frame is the moving Frenet frame.

According to Uzunoğlu et al. [15], another newly defined attractive frame is $\left\{N, C, W\right\}$ which serves as an alternate moving frame. By treating the principal normal vector N as a constant, such a frame represents the rotated state of a Frenet frame. It has a benefit over the Frenet frame in that the characterization of the slant helix may be expressed more succinctly with the new curvatures. The authors used the curve's curvature and torsion to establish new curvatures, f and g, while they were creating the $\{N,C,W\}$ frame. Regarding the strength of such new frame, they proposed the C-slant helix idea. Such kind of helix arises when the binormal and tangent indicatrix are both spherical slant helices. It was demonstrated that the C-constant precession curve is a C-slant helix and the C-slant helix's specific characterizations were developed. In the sense of the $\{N,C,W\}$ frame apparatus, they also provided the Frenet members of the indicatrix curves.

Our goal in this work is to investigate the combination of the $\{N,C,W\}$ frame, principal normal indicatrix curve and direction curves. We look into the characterizations of principal normal indicatrix curves' direction curves.

2. Basic Concept

Conte[illegible]fferentiable manifold M and an interval I, the differentiable function $\alpha: I \to M$ is referred to as the curve in differential geometry [12].

Three orthogonal vector fields are known as Frenet vectors for a regular curve α and $\alpha' \times \alpha'' \neq 0$. These vector fields are provided by

$$T = \frac{\alpha'}{\left\|\alpha'\right\|}, \quad \mathrm{N} = B \times T, \ \ \mathrm{B} = \frac{\alpha' \times \alpha''}{\left\|\alpha' \times \alpha''\right\|}. \tag{2.1}$$

The tangent vector T, the principal normal vector N and the binormal vector B are shown here. The curve's torsion and curvature are computed in the order of

$$\kappa = \frac{\left\|\alpha' \times \alpha''\right\|}{\left\|\alpha'\right\|^3}, \qquad \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\left\|\alpha' \times \alpha''\right\|^2} \tag{2.2}$$

and Frenet equations are valid as [12]

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N \end{aligned} \tag{2.3}$$

A new curve frame that follows a curve, the alternative $\{N,C,W\}$ frame, is described as

$$C = \frac{N'}{\left\| N' \right\|}, \qquad W = \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}}.$$

Here; N is the unit principal normal vector of Frenet frame, C is the derivative of principal normal vector and W is the unit Darboux vector. The new curvatures f and g with respect to this new frame are provided by

$$f = \kappa\sqrt{1 + H^2} \quad , \qquad g = \sigma f \tag{2.4}$$

where $H = \dfrac{\tau}{\kappa}$ isharmoniccurvatureofacurveand $\sigma = \dfrac{\kappa^2}{\left(\kappa^2 + \tau^2\right)^{3/2}}\left(\dfrac{\tau}{\kappa}\right)'$ is geodesic curvature of spherical image of principal normal indicatrix curve. According to Uzunoğlu et al. [15], the relationship between H and σ is also depicted as

$$\sigma = \frac{H'}{\kappa\left(1 + H^2\right)^{\frac{3}{2}}} \quad , \qquad \Gamma = \frac{\sigma'}{f\left(1 + \sigma^2\right)^{\frac{3}{2}}} . \tag{2.5}$$

Furthermore, when $H = \dfrac{\tau}{\kappa}$ is constant, a curve is referred to as a general helix, and the reverse is true [5]. Additionally, σ is constant if and only if it is a slant helix [6].

When a unit speed curve α has a vector field C that creates a constant angle $\theta \neq \dfrac{\pi}{2}$ with a fixed direction u , the curve is called a C-slant helix, in other words, $\left\langle C, u \right\rangle = \cos\theta = \text{constant}$ and the following equation holds for C-slant helix

$$\frac{\left(f^2 + g^2\right)^{\frac{3}{2}}}{f^2\left(\dfrac{g}{f}\right)'} = \tan\theta = \text{constant} \tag{2.6}$$

where the constant angle θ is formed between the vector C and the fixed direction u [15].

Now, same basic notions about indicatrix curves will be given.

When curve's each Frenet frame vector is translated to the center of the unit sphere and gathering the end points of them, then indicatrix curves are created on the surface of the unit sphere. The tangent, principal normal and binormal vectors constitute the tangent, principal normal and binormal indicatrix curves, respectively, and the corresponding equations for these curves are shown as [12]

$$\begin{aligned} \alpha_T &= T, \\ \alpha_N &= N, \\ \alpha_B &= B \end{aligned} \tag{2.7}$$

Uzunoğlu et al. [15] gave the equations between the Frenet members of the principal normal indicatrix curve α_N and the curve α, if $\{T_N, N_N, B_N, \kappa_N, \tau_N\}$ is Frenet apparatus of the principal normal indicatrix curve, as;

$$\kappa_N = \sqrt{\sigma^2 + 1}, \qquad \tau_N = \Gamma\sqrt{\sigma^2 + 1}. \tag{2.8}$$

and

$$\begin{aligned} T_N &= \frac{\kappa}{f}(-T + HB) \\ N_N &= \frac{\sigma}{\sqrt{\sigma^2+1}}\left[\frac{\kappa}{f}(HT+B) - \frac{1}{\sigma}N\right] \\ B_N &= \frac{1}{\sqrt{\sigma^2+1}}\left[\frac{\kappa}{f}(HT+B) + \sigma N\right]. \end{aligned} \tag{2.9}$$

The parametric curve that provides a unique solution to an equation system is known as an integral curve. If $\beta(t)$ is a parametric curve, Z is a vector field and if $\beta(t)$ solves the differential equation $\beta'(t) = Z(\beta(t))$, then $\beta(t)$ is referred to as an integral curve [12].

3. Direction Curves of Principal Normal Indicatrix Curve

This part includes the direction curves of principal normal indicatrix curves by using the apparatus of alternative frame $\{N, C, W\}$. Some characterizations and properties of these direction curves which are constructed by integrating the Frenet vectors of principal normal indicatrix curve are determined.

Let γ be a regular curve with Frenet frame $\{T, N, B\}$ and γ_N be the principal normal indicatrix curve of the curve γ. If the Frenet apparatus of γ_N is $\{T_N, N_N, B_N, \kappa_N, \tau_N\}$, then the following integral curves are called direction curves of principal normal indicatrix curve γ_N;

$$\begin{aligned} \phi_1 &= \int T_N, \\ \phi_2 &= \int N_N, \\ \phi_3 &= \int B_N \end{aligned} \tag{3.1}$$

Remark : Let s_1 and s_N be the arc length parameters of ϕ_1 and γ_N, respectively. Next, by differentiating and taking the norm the first equation in equation (3.1) with regard to s_1, we obtain $s_1 = s_N$. As a result, the arc length parameters of the principal normal indicatrix curve γ_N and the direction curves ϕ_1, ϕ_2 and $_3$ are equal.

Lemma 3.1: Let γ be a parametrized curve with arc length, $_N$ be principal normal indicatrix curve of γ and ϕ_1 be T_N - direction curve of γ_N. Frenet apparatus of the direction curve ϕ_1 is given as

$$T_{\phi_1} = \frac{\kappa}{f}\left(-T + HB\right),$$

$$N_{\phi_1} = \frac{\sigma\kappa H}{f\sqrt{\sigma^2+1}}T - \frac{1}{\sqrt{\sigma^2+1}}N + \frac{\sigma\kappa}{f\sqrt{\sigma^2+1}}B,$$

$$B_{\phi_1} = \frac{H}{\sqrt{(H^2+1)(\sigma^2+1)}}T + \frac{\sigma}{\sqrt{\sigma^2+1}}N + \frac{1}{\sqrt{(H^2+1)(\sigma^2+1)}}B, \tag{3.2}$$

$$\kappa_{\phi_1} = \sqrt{\sigma^2+1} \quad \text{and} \quad \tau_{\phi_1} = \Gamma\sqrt{(\sigma^2+1)} \quad .$$

Proof: The results are simply derived by taking the derivative of the first equation in (3.1), utilizing equations (2.1), (2.2), (2.3), (2.4) and (2.5) and performing the necessary calculations.

Corollary 3.1: Let γ be a parametrized curve with arc length, γ_N be principal normal indicatrix curve of γ and $_1$ be T_N - direction curve of γ_N. T_N -direction curve of γ_N and principal normal indicatrix curve of γ overlap.

Proof: Observing equations (2.8), (2.9) and (3.2), we see that the Frenet apparatus of γ_N and ϕ_1 are same. Aside from this, if we take derivative of second equation of (2.7) and first equation of (3.1), use Frenet formulas and $\frac{ds}{ds_N} = \frac{1}{f}$, the result is apparent.

Theorem 3.1: Let γ be a parametrized curve with arc length, γ_N be principal normal indicatrix curve of γ and ϕ_1 be T_N - direction curve of γ_N. If the curve γ is general helix, then T_N - direction curve of γ_N is planar.

Proof: If γ is general helix, then $H = \frac{\tau}{\kappa}$ is constant. So, by the equation (2.5), we have $\Gamma = 0$. According to equation (3.2), it is obvious that torsion of ϕ_1 is zero.

Theorem 3.2: Let γ be a parametrized curve with arc length, γ_N be principal normal indicatrix curve of γ and ϕ_1 be T_N - direction curve of γ_N. The curve γ is C-slant helix if and only if T_N - direction curve of γ_N is general helix.

Proof: Contemplating equations (2.4) and (2.5), the harmonic curvature of ϕ_1 is obtained as $H_{\phi_1} = \Gamma$. Also using equation (2.6), we have $H_{\phi_1} = \Gamma = \frac{1}{\tan\theta}$. Thus if T_N - direction curve of γ_N is general helix, then H_{ϕ_1} is constant. So $\tan\theta = \text{constant}$ and we have the result.

Theorem 3.3: Let γ be a parametrized curve with arc length, γ_N be principal normal indicatrix curve of γ and ϕ_1 be T_N - direction curve of γ_N. The curve γ is slant helix if and only if T_N - direction curve of γ_N is planar.

Proof: If the curve γ is slant helix, then σ is constant. So by the equation (2.5), we get $\Gamma = 0$. By using the equation (3.2), the torsion of ϕ_1 is zero. Also the proof is true for the opposite option.

Lemma 3.2: Let γ be a parametrized curve with arc length, γ_N be principal normal indicatrix curve of γ and ϕ_2 be N_N - direction curve of γ_N. Frenet apparatus of the direction curve ϕ_2 is given as

$$T_{\phi_2} = N_N = \frac{\sigma}{\sqrt{\sigma^2+1}}\left[\frac{\kappa}{f}(HT+B) - \frac{1}{\sigma}N\right],$$

$$N_{\phi_2} = \frac{\kappa\left(\sqrt{\sigma^2+1}+\Gamma H\right)}{f\sqrt{(\sigma^2+1)(\Gamma^2+1)}}T + \frac{\Gamma\sigma}{\sqrt{(\sigma^2+1)(\Gamma^2+1)}}N + \frac{\kappa\left(-\sqrt{\sigma^2+1}H+\Gamma\right)}{f\sqrt{(\sigma^2+1)(\Gamma^2+1)}}B,$$

$$B_{\phi_2} = \frac{H-\Gamma\sqrt{\sigma^2+1}}{\sqrt{(H^2+1)(\sigma^2+1)(\Gamma^2+1)}}T + \frac{\sigma}{\sqrt{(\sigma^2+1)(\Gamma^2+1)}}N + \frac{H\Gamma\sqrt{\sigma^2+1}+1}{\sqrt{(H^2+1)(\sigma^2+1)(\Gamma^2+1)}}B, \qquad (3.3)$$

$$\kappa_{\phi_2} = \sqrt{\left(\Gamma^2+1\right)\left(\sigma^2+1\right)} \quad \text{and} \quad \tau_{\phi_2} = \frac{\Gamma'}{f\left(\Gamma^2+1\right)}.$$

Proof: The results are simply derived by taking the derivative of the second equation in (3.1), utilizing equations (2.1), (2.2), (2.3), (2.4) and (2.5) and performing the necessary calculations.

Theorem 3.4: Let γ be a parametrized curve with arc length, γ_N be principal normal indicatrix curve of γ and ϕ_2 be N_N - direction curve of γ_N . If the curve γ is general helix, then N_N - direction curve of γ_N is planar.

Proof: If γ is general helix, then $H = \dfrac{\tau}{\kappa}$ is constant. So, by the equation (2.5), we have $\Gamma = 0$. Finally, the torsion of the curve ϕ_2 is zero, by considering equation (3.3).

Theorem 3.5: Let γ be a parametrized curve with arc length, γ_N be principal normal indicatrix curve of γ and ϕ_2 be N_N - direction curve of γ_N . The curve γ is C-slant helix if and only if N_N - direction curve of γ_N is planar.

Proof: Let the curve γ be C-slant helix. Then $\tan\theta = \text{constant}$, where θ is the angle between the vector field C and a fixed direction u. According to equations (2.4), (2.5) and (2.6) we have $\Gamma = \dfrac{1}{\tan\theta}$. Thus $\Gamma = \text{constant}$ and $\Gamma' = 0$. By the equation (3.3), the torsion of the curve ϕ_2 is zero and the result is clear.

Theorem 3.6: Let γ be a parametrized curve with arc length, γ_N be principal normal indicatrix curve of γ and ϕ_2 be N_N - direction curve of γ_N . N_N -direction curve can not be slant helix or general helix, if the curve γ is slant helix.

Proof: Considering equation (3.3), the harmonic curvature of the N_N -direction curve is obtained as

$$H_{\phi_2} = \frac{\Gamma'}{f\left(\Gamma^2+1\right)^{\frac{3}{2}}\sqrt{\left(\sigma^2+1\right)}}.$$

Also the σ function for being slant helix is

$$\sigma_{\phi_2} = \frac{1}{f}\frac{\left(\Gamma^2+1\right)\left(\sigma^2+1\right)}{\left(\left(\Gamma^2+1\right)\left(\sigma^2+1\right)+\left(\frac{\Gamma'}{f\left(\Gamma^2+1\right)}\right)^2\right)^{\frac{3}{2}}}\left(\frac{\Gamma'}{f\left(\Gamma^2+1\right)^{\frac{3}{2}}\sqrt{\left(\sigma^2+1\right)}}\right)'.$$

Here, σ is constant if the curve γ is slant helix. Therefore $H_{\phi_2}=0$ and $\sigma_{\phi_2}=0$, consequently, N_N -direction curve can not be general and slant helices.

Example: Lets consider arc length parametrized and general helix curve

$$\gamma(s)=\left(\frac{\cos^2(s)}{\sqrt{2}},-\frac{\sin(s)\cos(s)}{\sqrt{2}},\frac{s}{\sqrt{2}}\right).$$

The principal normal indicatrix curve of γ is

$\gamma_N(s)=\left(-\cos(2s),\sin(2s),0\right)$.

Then T_N -direction curve and N_N -direction curve of γ_N are found as

$$\phi_1(s)=\left(\frac{-\cos(2s)}{2}+c_1,\frac{\sin(2s)}{2}+c_2,c_3\right)$$

$$\phi_2(s)=\left(\frac{\sin(2s)}{2}+c_4,\frac{\cos(2s)}{2}+c_5,c_6\right)$$

where $c_i \in R,\quad 1 \le \mathrm{i} \le 9$.

It can be seen obviously that the curves ϕ_1 and ϕ_2 are planar, by calculating torsions of them. Thus we affirmed Theorem 3.1 and Theorem 3.4.

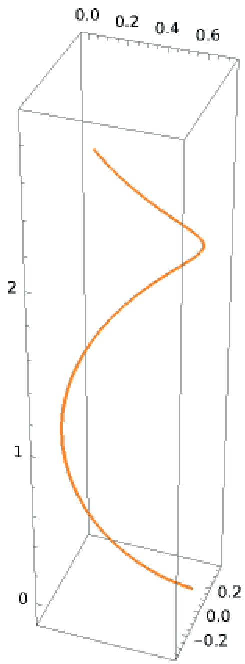

Figure 1. The curve γ.

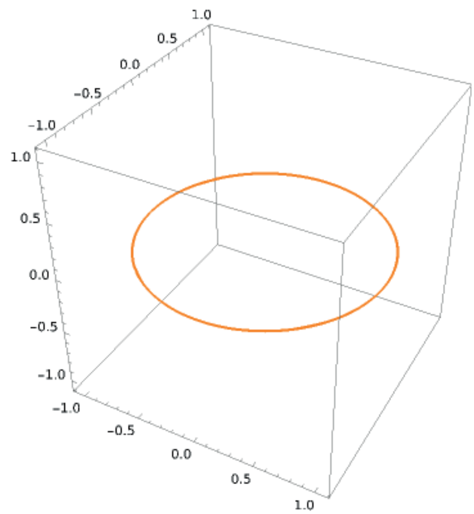

Figure 2. The curve γ_N.

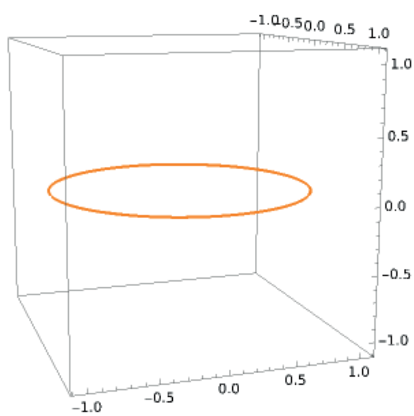

Figure 3. The curve ϕ_1.

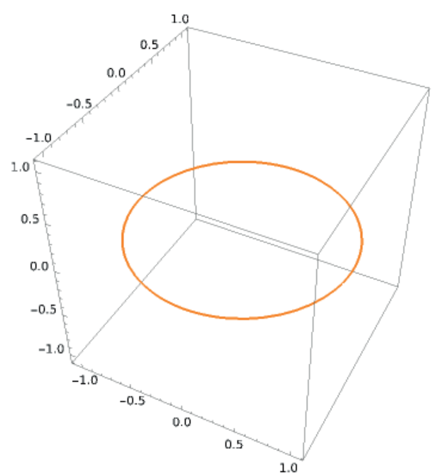

Figure 4. The curve ϕ_2.

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