Chapter 11

Structural and Statistical Analysis of Finite Mixture Models Based on q-Calculus

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Abstract

The foundations of q -analysis date back to the 1740s, when Euler introduced the theory of partitions, also referred to as additive analytic number theory. Over the years, the discovery of q -calculus applications in fields such as operator theory, combinatory, probability theory, and many others has sparked tremendous interest in this mathematical framework.

Mixture distributions are probabilistic models in which a data set is assumed to originate from multiple underlying distributions, each contributing with a certain probability. These distributions are commonly used to model complex data structures more accurately.

This paper introduces q -finite mixture models as a novel extension of the classical finite mixture family, motivated by recent progress in q -calculus and generalized probability distributions. By incorporating a deformation parameter q , the proposed mixture models offer enhanced modeling flexibility for a variety of stochastic phenomena. The fundamental distributional and statistical properties of the suggested q -mixture models are systematically are explored.

INTRODUCTION

Quantum calculus, also known as q -calculus or calculus without limit is a generalization of classical calculus that originated in the early 20th century, although its roots can be traced even further back. Euler studied the q -analog of Newton's infinite series and made foundational contributions. Jacobi formulated the Gauß-Jacobi triple product identity. Gauß introduced the q -binomial coefficients and established identities involving them. Jackson defined the concept of the q -integral. Ernst provided a comprehensive

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historical overview and proposed a new approach to q-calculus. Cheung and Kac authored the monograph Quantum Calculus, further developing the field.

Mixture distributions are probabilistic models in which observations are assumed to originate from multiple underlying distributions, each with a certain probability. The evolution of q-distributions represents a natural progression in the development of q-calculus. q-calculus serves as a parametric generalization of classical calculus, with the classical framework being recovered in the limit as $q \to 1$. Significant contributions to the theory of q-distributions including such as Dunkl, 1981, Crippa et al., 1997, Kupershmidt, 2000, Kemp, 2002, Charalambides, 2016, including the Gaussian and generalized gamma q-distributions by Diaz et al., 2009, 2010, the Erlang q-distributions by Charalambides, 2016, the gamma and beta q-distributions by Boutouria et al., 2018, the Lindley q-distribution in two forms was introduced by Bouzida, 2023.

In response to recent progress in the study of generalized probability q-distributions, this paper presents q-finite mixture distribution with their fundamental statistical and distributional characteristics.

MATERIAL AND METHODS

This section outlines the principles of q-calculus, and q-probability theory. In this entire study, unless otherwise stated, it is assumed that $0 < q < 1$. Readers are referred to the relevant literature.

Definition 1 (Kac and Cheung, 2002). Let x, q be real numbers. The q-number $[x]_q$ is defined as

$$[x]_q = \frac{1-q^x}{1-q}.$$

Definition 2 (Kac and Cheung, 2002). The q-Gauss binomial formula is given by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} y^k x^{n-k}, \quad -\infty < x, y < \infty.$$

The q-binomial coefficients are provided for $k = 0, 1, \ldots, n$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}, \quad [n]_{k,q} = \frac{[n]_q!}{[n-k]_q!}, \quad [n]_q! = [n]_q [n-1]_q \ldots [2]_q [1]_q.$$

Definition 3 (Kac and Cheung, 2002). The q-analogues of the exponential function are presented

$$E_q^\tau = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{\tau^k}{[k]_q!} = \prod_{k=0}^{\infty}\left(1+(1-q)q^k\tau\right), \quad \tau \in \mathbb{R},$$

$$e_q^\tau = \sum_{k=0}^{\infty} \tau = \prod_{k=0}^{\infty} \frac{1}{\left(1-(1-q)q^k\tau\right)}, \quad |\tau| < \frac{1}{1-q}.$$

Definition 4 (Kac and Cheung, 2002). The q-derivative of f is defined as

$$D_q f(\tau) = \frac{f(q\tau)-f(\tau)}{q\tau-\tau}$$

Definition 5 (Kac and Cheung, 2002). The well-known Jackson q-integral of f is given by

$$\int_0^b f(\tau)\, d_q\tau = (1-q)\sum_{n=0}^{\infty} q^n b f\left(q^n b\right),\ b > 0.$$

Definition 6 (Vamvakari, 2023). X is considered q-continuous if there exists $f_q^X(x)$ such that

$$P\{a < X \le b\} = \int_a^b f_q^X(x)\, d_q x,\ x \ge 0.$$

The q-cumulative distribution function (q-CDF) of X is defined for $x > 0$

$$F_q^X(x) = P(X \le x) = \int_0^x f_q^X(u)\, d_q u,$$

satisfying the relation $P(\alpha < X \le \beta) = F_q^X(\beta) - F_q^X(\alpha)$. Then, $f_q(x) = D_q F_q(x)$.

Definition 7 (Vamvakari, 2023). Under the condition $\xi_{(1)} < q\xi_{2)} < \xi_{(2)} < \ldots < \xi_{(n-1)} < q\xi_{(n)}$ the random variable $\xi_{(\nu)}$ is defined as ν-th q-ordered random variable. The q-CDFs of the q-ordered statistics $\xi_{(n)}$, $\xi_{(1)}$, $\xi_{(\nu)}$ ($1 \le \nu \le n$) are expressed, respectively

$$F_q^{\xi_{(n)}}(\tau)=\prod_{i=1}^{n}F_q\left(q^{i-1}\tau\right),\ \ F_q^{\xi_{(1)}}(\tau)=1-\prod_{i=1}^{n}\left(1-F_q(\tau)\right),$$

$$F_q^{\xi_{(v)}}(\tau)=\sum_{w=v}^{n}\sum_{1<i_1<\ldots<i_w<n}\prod_{j=1}^{r}F_q\left(q^{j-1}\tau\right)\prod_{m=w+1}^{n}\left(1-F_q\left(q^{i_m-(m-w)}\tau\right)\right)$$

Definition 8 (Vamvakari, 2023). Let $\mathbb{E}_q\left|\xi^r\right|<\infty$ for all positive integers r. Then,

$$\mu_q^{(r)}=\mathbb{E}_q\left(\xi^r\right)=\int_0^{\infty}\tau^r f_q^{\xi}(\tau)\,d_q\tau,\ \ \mathbb{E}_q(\xi)=\mu_q,\ \ \mathbb{V}_q(\xi)=\mu_q^{(2)}-\left(\mu_q\right)^2.$$

Definition 9 (Okur and Djongmon, 2025). Let ξ be a q-continuous non-negative RV, and $\mathbb{E}_q\left|\left(\xi-\mu_q\right)_q^r\right|<\infty$ for all positive integers r. Then,

$$m_q^{(r)}=\mathbb{E}_q\left(\xi-\mu_q\right)_q^r=\sum_{s=0}^{r}\begin{bmatrix} r \\ s \end{bmatrix}_q q^{\binom{s}{2}}(-1)^s\,\mu_q^s\mu_q^{(r-s)},\ \ s\le r.$$

Definition 10 (Okur and Djongmon, 2025). Let ξ be a q-continuous non-negative RV. Then, its q-MGF is expressed in two distinct forms as follows:

$$\mathbb{M}_q^{I}(t)=\mathbb{E}_q\left(E_q^{qt\xi}\right)=\int_0^{\infty}E_q^{qt\tau}f_q(\tau)\,d_q\tau,\ \mathbb{M}_q^{II}(t)=\mathbb{E}_q\left(e_q^{t\xi}\right)=\int_0^{\infty}e_q^{t\tau}f_q(\tau)\,d_q\tau.$$

Definition 11 (Djongmon and Okur, 2025). Let ξ be a q-continuous non-negative RV. Then,

the q-survival function (q-CCDF) $\mathbb{S}_q(\tau)=P(\xi>\tau)=1-F_q(\tau)$,

the q-hazard rate function (q-HRF)

$$h_q(\tau)=\frac{P\left(q\tau\le\xi\le\tau\,\middle|\,\xi\ge q\tau\right)}{(1-q)\tau}=\frac{f_q(\tau)}{\mathbb{S}_q(q\tau)},$$

the q-mean residual life function (q-MRLF)

$$mrl_q(\tau)=\mathbb{E}_q\left(\xi-\tau\,\middle|\,\xi>\tau\right)=\frac{1}{\mathbb{S}_q(\tau)}\int_{\tau}^{\infty}\mathbb{S}_q(qu)\,d_qu.$$

Definition 12 (Djongmon and Okur, 2025). Let X and Y be two independent q-continuous non-negative random variables. The q-stress-strenght reliability (q-SSR) is given by:

$$\mathbb{R}_q = P(X > Y) = \int_0^{\infty} P(X > \tau | Y = \tau) f_q^X(\tau) d_q\tau = \int_0^{\infty} f_q^X(\tau) \mathrm{F}_q^Y(\tau) d_q\tau.$$

RESULTS

This section outlines a q-finite mixture model, including structral and statistical properties.

Modeling of a q-Finite Mixture Model

A q-finite mixture model (denoted q-FMM) is a q-probabilistic model composed of multiple component q-distributions combined with certain weights. The general form of mixture q-PDF is described as:

$$f_q^{mix(i_m)}(x) \quad = \sum_{k=1}^{K} \pi_{q_i}^{(k)} \cdot f_{q_i}^{C_m(k)}\left(x; \lambda^{(k)}\right),\ q_i \in \{q, 1/q\},\ 0 < q < 1$$

where:

$i \in \{I, II\}$: type of q-mixture model such that q, for $i = I$ and $1/q$, for $i = II$

$m \in \{1, 2, \cdots, 2^n\}$: number of q-mixture model

K : number of components,

$\pi_{q_i}^{(k)}$: mixing proportion of the k-th component, $\pi_{q_i}^{(k)} \geq 0$, $\sum_{k=1}^{n} \pi_{q_i}^{(k)} = 1$

C_m : selecting function the appropriate q or $1/q$-PDF for the k-th density component

$\lambda_{(k)}$: parameter vector of the k-th density component

$f_q^{(k)}\left(x; \lambda^{(k)}\right)$: the k-th q-density component

$f_{1/q}^{(k)}\left(x; \lambda^{(k)}\right)$: the k-th $1/q$-density component

As $q \to 1$, the q-finite mixture model converges to its ordinary form.

Structure Properties of the q-Finite Mixture Model

The q-finite mixture model represents a general modeling framework that subsumes both q-homogeneous and q-hybrid variants, offering enhanced flexibility for representing heterogeneous systems characterized by different q-parametrizations. Let $\mathbb{I}_A(k)$ be the indicator function equal to 1 if $k \in A$, and 0 otherwise, and $\mathbb{I}_B(k)$ is similarly defined for complementary set $B \subset \{1,2,\ldots,K\}$, where $A \cup B = \{1,2,\ldots,K\}$.

q-Homogeneous Finite Mixture Model. A q-homogeneous mixture model refers to a class of mixture models in which all component distributions are derived exclusively from a single formulation family—either the original q-formulation or its reciprocal counterpart based on $1/q$. The homogeneity of the component structure ensures a consistent parametric behavior across the mixture, which is particularly advantageous for analytical tractability and interpretability within the same functional family. Hence, it can be formulated:

$$f_q^{mix(i_m)}(x) = \sum_{k=1}^{K}\left(\pi_q^{(k)} \cdot \mathbb{I}_A(k) + \pi_{1/\mathrm{q}}^{(k)} \cdot \mathbb{I}_B(k)\right) \cdot f_{q_i}^{(k)}\left(x;\lambda^{(k)}\right).$$

q-Hybrid Finite Mixture Model. A q-hybrid mixture model is a type of mixture model that incorporates component distributions originating from both the original q-formulation and its reciprocal counterpart based on $1/q$. Unlike the q-homogeneous model, the q-hybrid structure allows for the coexistence of multiple generative mechanisms within a single framework. Thus, it can be defined as:

$$f_q^{mix(i_m)}(x) = \sum_{k=1}^{K}\pi_{q_i}^{(k)} \cdot \left(f_q^{(k)}\left(x;\lambda^{(k)}\right) \cdot \mathbb{I}_A(k) + f_{1/q}^{(k)}\left(x;\lambda^{(k)}\right) \cdot \mathbb{I}_B(k)\right).$$

Statistical Properties of the q-Finite Mixture Model

The corresponding q-mixture statistical characteristics are given by

q-cumulative distributional function of model and the q-ordered statistics

$$F_q^{mix(i_m)}(x) = \sum_{k=1}^{K}\pi_{q_i}^{(k)} \cdot F_{q_i}^{C_m(k)}\left(x;\lambda^{(k)}\right), F_q^{\xi_{(1)}(mix(i_m))}(x) = \sum_{k=1}^{K}\pi_{q_i}^{(k)} \cdot F_{q_i}^{\xi_{(1)}(C_m(k))}\left(x;\lambda^{(k)}\right)$$

$$F_q^{\xi_{(n)}(mix(i_m))}(x) = \sum_{k=1}^{K}\pi_{q_i}^{(k)} \cdot F_{q_i}^{\xi_{(n)}(C_m(k))}\left(x;\lambda^{(k)}\right), F_q^{\xi_{(v)}(mix(i_m))}(x) = \sum_{k=1}^{K}\pi_{q_i}^{(k)} \cdot F_{q_i}^{\xi_{(v)}(C_m(k))}\left(x;\lambda^{(k)}\right)$$

q-moment, q-central moment, q-expectation and q-variance:

$$\mu_q^{(r)(mix(i_m))} = \sum_{k=1}^{K}\pi_{q_i}^{(k)} \cdot \mu_{q_i}^{(r)(C_m(k))},\ m_q^{(r)(mix(i_m))} = \sum_{k=1}^{K}\pi_{q_i}^{(k)} \cdot m_{q_i}^{(r)(C_m(k))}$$

$$\mathbb{E}_q^{mix(i_m)}\left(\xi\right)=\sum_{k=1}^{K}\pi_{q_i}^{(k)}\cdot\mathbb{E}_{q_i}^{C_m(k)}\left(\xi^{(k)}\right),\ \mathbb{V}_q^{mix(i_m)}\left(\xi\right)=\sum_{k=1}^{K}\pi_{q_i}^{(k)}\cdot\mathbb{V}_{q_i}^{C_m(k)}\left(\xi^{(k)}\right)$$

q-moment generating function:

$$\mathbb{M}_q^{I(mix(i_m))}=\sum_{k=1}^{K}\pi_{q_i}^{(k)}\cdot\mathbb{M}_{q_i}^{I(C_m(k))}\left(t\right),\ \mathbb{M}_q^{II(mix(i_m))}=\sum_{k=1}^{K}\pi_{q_i}^{(k)}\cdot\mathbb{M}_{q_i}^{II(C_m(k))}\left(t\right)$$

q-reliability functions and q-stress-strength reliability:

$$\mathbb{S}_q^{mix(i_m)}\left(x\right)=\sum_{k=1}^{K}\pi_{q_i}^{(k)}\cdot\mathbb{S}_{q_i}^{C_m(k)}\left(x;\lambda^{(k)}\right),\ h_q^{mix(i_m)}\left(x\right)=\sum_{k=1}^{K}\pi_{q_i}^{(k)}\cdot h_{q_i}^{C_m(k)}\left(x;\lambda^{(k)}\right)$$

$$mrl_q^{mix(i_m)}\left(x\right)=\sum_{k=1}^{K}\pi_{q_i}^{(k)}\cdot mrl_{q_i}^{C_m(k)}\left(x;\lambda^{(k)}\right),\ \mathbb{R}_q^{mix(i_m)}=\sum_{k=1}^{K}\pi_{q_i}^{(k)}\cdot\mathbb{R}_{q_i}^{C_m(k)}$$

An Illustirative Example (q-Exponential Finite Mixture Model)

A q-exponential finite mixture model (q-EFMM) is constructed by combining multiple q-exponential distributional components through a weighted linear combination. These weights reflect the relative contributions of each component and may be generalized using the q-algebra framework. The resulting model provides a flexible representation for heavy-tailed data. Its general form is:

$$\xi\sim\sum_{k=1}^{K}\pi_{q_i}^{(k)}\cdot\mathrm{Exp}_q\left(x;\lambda^{(k)}\right),\ q_i\in\left\{q,1/q\right\},\ 0<q<1$$

where $\mathrm{Exp}_q\left(x;\lambda^{(k)}\right)$ represents th exponential q-distribution and the corresponding q and $1/q$-component densities are:

$$f_q^{(k)}\left(x;\lambda^{(k)}\right)=\lambda^{(k)}E_q^{-q\lambda^{(k)}x}\cdot\mathbb{I}_{[0,1/1-\mathrm{q}]}\left(x\right),$$

$$f_{1/q}^{(k)}\left(x;\lambda^{(k)}\right)=\lambda^{(k)}e_q^{-\lambda^{(k)}x}\cdot\mathbb{I}_{[0,\infty)}\left(x\right).$$

For $K = 2$, let us describe two types of the q-mixture exponential model with a fixed mixing proportion. For the sake of simplification, let

$$\pi_q^{(1)} = \omega_q, \pi_q^{(2)} = 1 - \omega_q \ , \ \pi_{1/q}^{(1)} = \omega_{1/q}, \pi_{1/q}^{(2)} = 1 - \omega_{1/q}$$

Thus, the parameters of the q-density components may either be identical or non-identical. In such a case, $\lambda^{(1)} = \lambda^{(2)} = \lambda > 0$ or alternatively, $\lambda^{(1)} = \alpha \neq \lambda^{(2)} = \beta$, where $\alpha, \beta > 0$. Accordingly, the distributional characteristics of the q-exponential mixture model (q-EMM) are given by:

Table 1. The exponential mixture q-PDFs with identical component parameters

m	The Homogeneous Mixture q-PDF	The Hybrid Mixture q-PDF
1	$\lambda E_q^{-q\lambda x}$	$\omega_q \lambda E_q^{-q\lambda x} + \left(1 - \omega_q\right) \lambda e_q^{-\lambda x}$
2	$\lambda e_q^{-\lambda x}$	$\omega_q \lambda e_q^{-\lambda x} + \left(1 - \omega_q\right) \lambda E_q^{-q\lambda x}$
3	$\lambda E_q^{-q\lambda x}$	$\omega_{1/q} \lambda E_q^{-q\lambda x} + \left(1 - \omega_{1/q}\right) \lambda e_q^{-\lambda x}$
4	$\lambda e_q^{-\lambda x}$	$\omega_{1/q} \lambda e_q^{-\lambda x} + \left(1 - \omega_{1/q}\right) \lambda E_q^{-q\lambda x}$

Table 2. The exponential mixture q-PDFs with non-identical component parameters

m	The Homogeneous Mixture q-PDF	The Hybrid Mixture q-PDF
1	$\omega_q \alpha E_q^{-q\alpha x} + \left(1 - \omega_q\right) \beta E_q^{-q\beta x}$	$\omega_q \alpha E_q^{-q\alpha x} + \left(1 - \omega_q\right) \beta e_q^{-\beta x}$
2	$\omega_q \alpha e_q^{-\alpha x} + \left(1 - \omega_q\right) \beta e_q^{-\beta x}$	$\omega_q \alpha e_q^{-\alpha x} + \left(1 - \omega_q\right) \beta E_q^{-q\beta x}$
3	$\omega_{1/q} \alpha E_q^{-q\alpha x} + \left(1 - \omega_{1/q}\right) \beta E_q^{-q\beta x}$	$\omega_{1/q} \alpha E_q^{-q\alpha x} + \left(1 - \omega_{1/q}\right) \beta e_q^{-\beta x}$
4	$\omega_{1/q} \alpha e_q^{-\alpha x} + \left(1 - \omega_{1/q}\right) \beta e_q^{-\beta x}$	$\omega_{1/q} \alpha e_q^{-\alpha x} + \left(1 - \omega_{1/q}\right) \beta E_q^{-q\beta x}$

The graphs below illustrate the q-exponential mixture model with identical component parameters for $\omega = 0.75$ and $\lambda = 2$, with varying values of the q-parameter as outlined in Table 1.

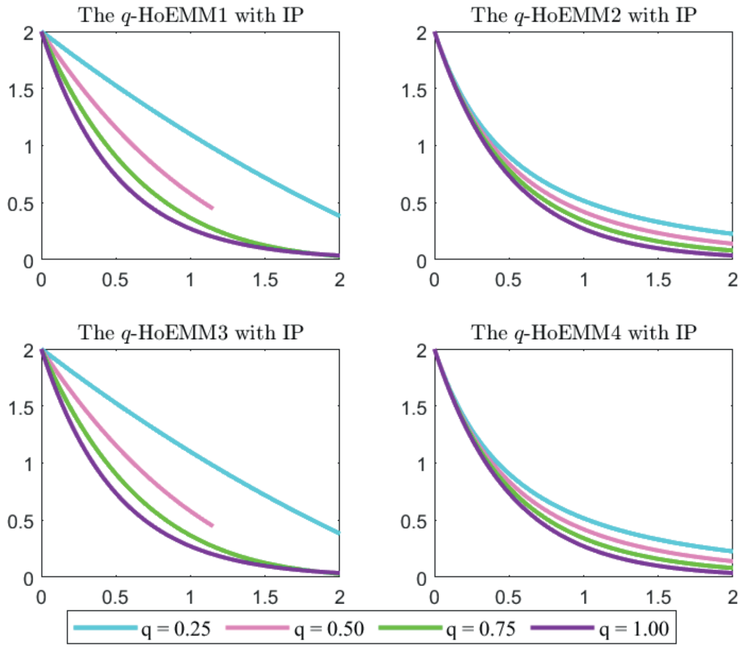

Figure 1. Graph of the homogeneous exponential mixture q ***-PDFs with identical component parameters for*** $\omega = 0.75$ ***and*** $\lambda = 2$

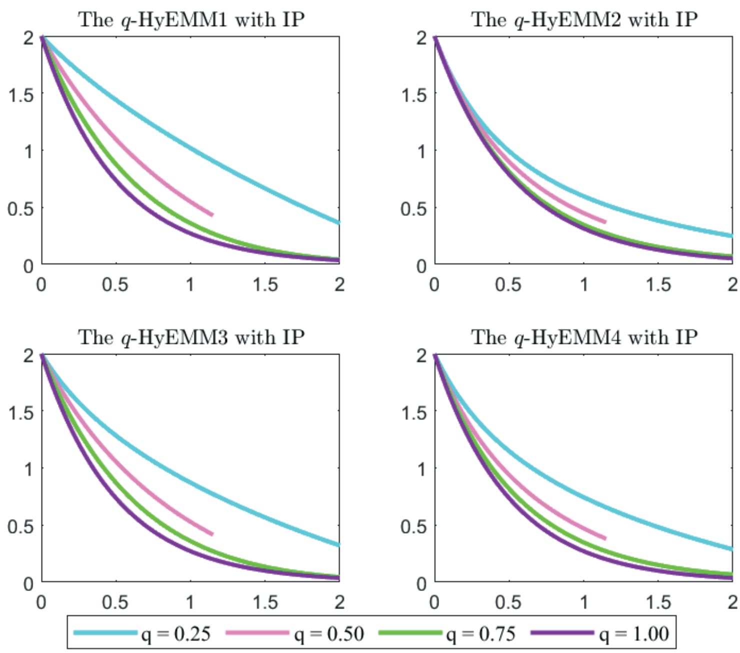

Figure 2. Graph of the hybrid exponential mixture q ***-PDFs with identical component parameters for*** $\omega = 0.75$ ***and*** $\lambda = 2$

Presented below is the graph of the q-exponential mixture model with non-identical component parameters corresponding to $\omega = 0.75$ and $\alpha = 1, \beta = 3$, under varying q -parameter values as specified in Table 2. The homogeneous and hybrid configurations are depicted in Figures 3 and 4:

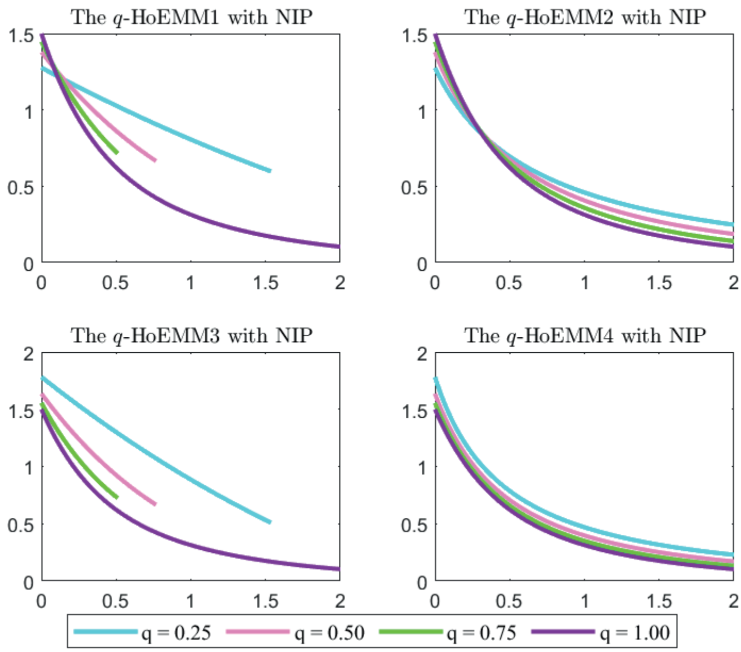

Figure 3. Graph of the homogeneous exponential mixture q -PDFs with non-identical component parameters for $\omega = 0.75$ and $\alpha = 1, \beta = 3$

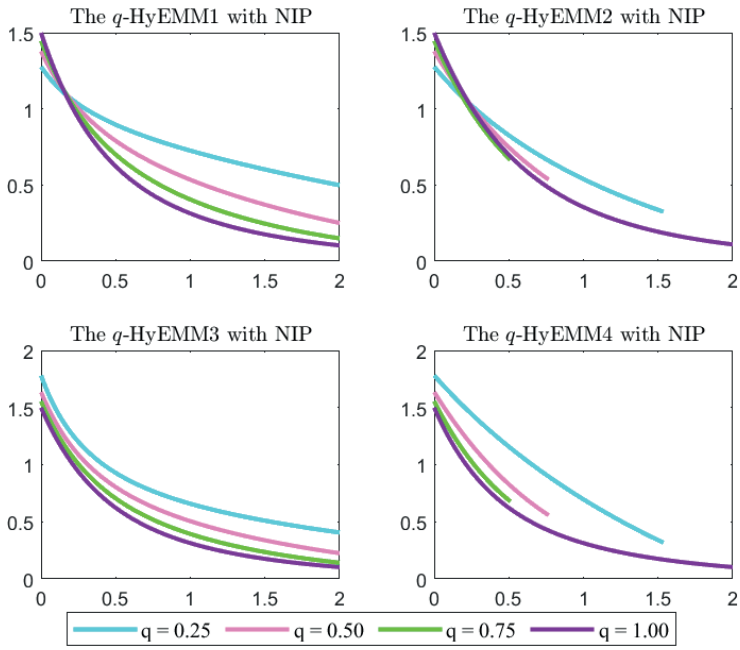

Figure 4. Graph of the hybrid exponential mixture q-PDFs with non-identical component parameters for $\omega = 0.75$ and $\alpha = 1, \beta = 3$

Above, q-HoEMMm $(m = 1-4)$ and q-HyEMMm $(m = 1-4)$ denote homogeneous and hybrid q-exponential mixture models, respectively. NIP and IP stand for non-identical and identical parameters. The plots show that increasing q enhances model convexity, and all q-exponential mixtures reduce to the standard form as $q \to 1$.

DISCUSSION AND CONCLUSION

Probability q-distributions provide a flexible and dynamic framework that generalizes classical probability distributions by introducing the q-parameter. This parameter allows for a broader class of probabilistic models, enriching both theoretical understanding and practical applications.

In this paper, we introduce q-finite mixture model, and and provide a detailed analysis of the structural and statistical properties. As a representative example to elucidate the core concept, this study presents an exponential mixture model with a fixed mixing proportion, along with a discussion of its properties. Our findings suggest that the proposed q-distribution holds significant promise and may have widespread applications across various fields. In future research, we aim to explore the finite mixture and compound q-distribution.

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Conflict of Interest

The authors have declared that there is no conflict of interest.

Author Contributions

Both authors contributed equally to the finalization of this paper.