Chapter 14

Homogeneous and Hybrid *q*-Mixture Forms of the Uma Distribution

Nurgül Okur[1]

Hasan Hüseyin Gül[2]

Abstract

Probability *q*-distributions provide a flexible and dynamic framework that generalizes classical probability distributions by introducing the *q*-parameter. This parameter allows for a broader class of probabilistic models, enriching both theoretical understanding and practical applications.

This study presents a comprehensive framework for the *q*-analogues of the Uma distribution and provides an in-depth analysis of the specific cases corresponding to the homogeneous and hybrid types. Although these distributions exhibit similar probability density functions, they exhibit significant differences in terms of parameter constraints, generative mechanisms, and underlying statistical properties.

The study encompasses the modeling of the *q*-analogues of both the probability density function and the cumulative distribution function, along with a detailed exploration of their shapes through rigorous mathematical analysis. Furthermore, the fundamental statistical and distributional characteristics of these distributions are thoroughly examined. Ultimately, these findings offer valuable insights for the continued exploration and application of *q*-Uma distributions across various fields.

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1 Giresun University, Faculty of Science and Arts, Department of Data Science and Analytics, Türkiye

2 Giresun University, Faculty of Science and Arts, Department of Data Science and Analytics, Türkiye

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INTRODUCTION

Over the past few years, there has been a significant rise in the proposal of lifetime distributions for modeling lifetime data by various statisticians. In this context, the Uma distribution was introduced by Shanker in 2017, along with its probability density function (PDF) and cumulative distribution function (CDF), for the parameters $x \geq 0$ and $\lambda > 0$:

$$f(x;\lambda) = a\left(1+x+x^3\right)e^{-\lambda x},\ a = \frac{\lambda^4}{\lambda^3+\lambda^2+3!},$$

$$F(x;\lambda) = 1-\left\{1+\frac{a}{\lambda^4}\left[(\lambda x)^3+3(\lambda x)^2+\left(\lambda^2+3!\right)\lambda x\right]\right\}e^{-\lambda x}.$$

In probability theory, the concept of q-distributions expands upon classical distributions, offering a wider array of possibilities. Key contributions to the theory of basic discrete q-distributions include the works of Dunkl, 1981, Crippa et al., 1997, Kupershmidt, 2000, Kemp, 2002, Charalambides, 2016, Kyriakoussis and Vamvakari, 2017, among others. Significant research on q-continuous distributions, including the Gaussian and generalized gamma q-distributions by Diaz et al., 2009, 2010, the Erlang q-distributions by Charalambides, 2016, the gamma and beta q-distributions by Boutouria et al., 2018, the Lindley q-distribution in two forms was introduced by Bouzida, 2023.

This research develops a comprehensive framework for the q-analogues of the Uma distribution, along with an extensive analysis of the particular cases that correspond to the homogeneous and hybrid types.

MATERIAL AND METHODS

This section outlines the principles of q-calculus, and q-probability theory. In this entire study, unless otherwise stated, it is assumed that $0 < q < 1$. Readers are referred to the relevant literature.

Definition 1 (Kac and Cheung, 2002). Let x, q be real numbers. The q-number $[x]_q$ is defined as

$$[x]_q = \frac{1-q^x}{1-q}.$$

For $n \in \mathbb{N}$, a natural q-number, it reduces to $[n]_q = \sum_{k=0}^{n-1} q^k$.

Definition 2 (Kac and Cheung, 2002). The q-Gauss binomial formula is given by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} y^k x^{n-k}, \quad -\infty < x, y < \infty.$$

The q-binomial coefficients are provided for $k = 0,1,\ldots,n$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}, \; [n]_{k,q} = \frac{[n]_q!}{[n-k]_q!}, \; [n]_q! = [n]_q[n-1]_q \ldots [2]_q[1]_q.$$

Definition 3 (Kac and Cheung, 2002). The q-analogues of the exponential function are presented

$$E_q^\tau = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{\tau^k}{[k]_q!} = \prod_{k=0}^{\infty}\left(1+(1-q)q^k\tau\right), \quad \tau \in \mathbb{R},$$

$$e_q^\tau = \sum_{k=0}^{\infty} \frac{\tau^k}{[k]_q!} = \prod_{k=0}^{\infty} \frac{1}{\left(1-(1-q)q^k\tau\right)}, \; |\tau| < \frac{1}{1-q}.$$

Definition 4 (Kac and Cheung, 2002). The q-derivative of f is defined as

$$D_q f(\tau) = \frac{f(q\tau)-f(\tau)}{q\tau-\tau}.$$

Definition 5 (Kac and Cheung, 2002). The well-known Jackson q-integral of f is given by

$$\int_0^b f(\tau)\,d_q\tau = (1-q)\sum_{n=0}^{\infty} q^n b f(q^n b), \; b>0,$$

Definition 6 (Kac and Cheung, 2002). The generalized q-integral is given by

$$\int_{-\infty}^{\infty} f(\tau)\,d_q\tau = (1-q)\sum_{n\in\mathbb{Z}} q^n f(q^n).$$

Definition 7 (Kac and Cheung, 2002, De Sole and Kac, 2003). The q-analogues of the gamma functions are given for $\alpha > 0$, $n \in \mathbb{N}$

$$\Gamma_q(\alpha) = \int_0^{[\infty]_q} x^{\alpha-1} E_q^{-qx} d_q x, \; \gamma_q(\alpha) = \int_0^{\infty} x^{\alpha-1} e_q^{-x} d_q x.$$

Identities derived from the q-gamma functions can be obtained

$$\Gamma_q(\alpha+n)=[\alpha]_{n,q}\Gamma_q(\alpha) \qquad \gamma_q(\alpha+n)=q^{-\binom{\alpha+n}{2}}[\alpha]_{n,q}\gamma_q(\alpha)$$

$$\Gamma_q(n+1)=[n]_q! \qquad \gamma_q(n+1)=q^{-\binom{n+1}{2}}[n]_q!$$

Definition 8 (Vamvakari, 2023). A random variable X is considered q-continuous if there exists a non-negative function $f_q^X(x)$ for $x\geq 0$ such that

$$P\{a<X\leq b\}=\int_a^b f_q^X(x)d_qx.$$

The q-cumulative distribution function (q-CDF) of X. is defined for $x>0$

$$F_q^X(x)=P(X\leq x)=\int_0^x f_q^X(u)d_qu,$$

satisfying the relation $P(\alpha<X\leq\beta)=F_q^X(\beta)-F_q^X(\alpha)$. Then,

$$f_q(x)=D_qF_q(x)=\frac{F_q^X(x)-F_q^X(qx)}{(1-q)x}=\frac{P(qx\leq X\leq x)}{(1-q)x},\ (q\neq 1\ ).$$

Definition 9 (Vamvakari, 2023). Let $\mathbb{E}_q\left|X^r\right|<\infty$ for all positive integers r. Then,

$$\mu_q^{(r)}=\mathbb{E}_q\left(X^r\right)=\int_0^\infty x^r\ f_q^X(x)d_qx,\ \mathbb{E}_q(X)=\mu_q,\ \mathbb{V}_q(X)=\mu_q^{(2)}-\mu_q^2.$$

RESULTS

This section presents the Uma q-distribution, including its homogeneous and hybird forms.

Modeling of the Uma q-distribution

A q-mixture model of the Uma distribution is a q-probabilistic parametric model composed of multiple component Uma$q$-distributions combined with certain parametric weights. The general form of the q-parametric mixture model of the Uma distribution is for $x\geq 0$. characterized by

$$f_q^{i_n}(x;\lambda)=\sum_{\alpha\in D}p_q^i(\alpha)g_q^{C_n}(x;\alpha,\lambda),\ (i=I,II;\ n=1,2,...,8)$$

where

$\alpha \in D$: Parameter value in the set of components $D=\{1,2,4\}$

$p_q^i(\alpha)$: Mixing parametric proportion of the α-th component, $p_q^i(\alpha)\geq 0$, $\sum_{\alpha\in D} p_q^i(\alpha)=1$ and

$$p_q^I(\alpha)=\frac{\Gamma_q(\alpha)\lambda^{1-\alpha}}{\sum_{k\in D}\Gamma_q(\alpha)\lambda^{1-\alpha}},\ p_q^{II}(\alpha)=\frac{\gamma_q(\alpha)\lambda^{1-\alpha}}{\sum_{k\in D}\gamma_q(\alpha)\lambda^{1-\alpha}}$$

$g_q^{C_n}(x;\alpha,\lambda)$: Gamma q-PDF of the α-th component and

$$g_q^I(x;\alpha,\lambda)=\frac{\lambda^\alpha}{\Gamma_q(\alpha)}x^{\alpha-1}E_q^{-q\lambda x}1_{\left[0,[\infty]_q\right]}(x),\ g_q^{II}(x;\alpha,\lambda)=\frac{\lambda^\alpha}{\gamma_q(\alpha)}x^{\alpha-1}e_q^{-\lambda x}1_{[0,\infty)}(x)$$

C_n : Selecting function the appropriate q or $1/q$-PDF.

As $q\to 1$, the q-PDF $f_q^{i_n}$ converges to the ordinary Uma PDF. The function $f_q^{i_n}$ satisfies the necessary conditions to qualify as a q-PDF. Furthermore, the corresponding q-CDF is:

$$F_q^{i_n}(x;\lambda)=\sum_{\alpha\in D}p_q^i(\alpha)G_q^{C_n}(x;\alpha,\lambda),\ (i=I,II;\ n=1,2,\ldots,8)$$

where $G_q^i(x,\alpha,\lambda)$ is the gamma q-CDF of the α-th component, and

$$G_q^I(x,\alpha,\lambda)=\frac{\Gamma_q(\alpha,\lambda x)}{\Gamma_q(\alpha)},\ G_q^{II}(x,\alpha,\lambda)=\frac{\gamma_q(\alpha,\lambda x)}{\gamma_q(\alpha)}$$

The q-parametric mixture model of the Uma distribution is a broad family of distributions that includes the original q-formulation, the reciprocal formulation based on $1/q$, as well as their hybrid combinations. Let $p_q^i(\alpha)$ and $g_q^i(x;\alpha,\lambda)$ be $p_q^{i(\alpha)}$ and $g_q^{i(\alpha)}$, and let us define the notations:

$$a_q^I=\frac{\lambda^4}{\lambda^3+\lambda^2+[3]_q!},\ a_q^{II}=\frac{\lambda^4}{\lambda^3+q^{-1}\lambda^2+q^{-6}[3]_q!}$$

Homogeneous Type of the Uma q-distribution

This section introduces the homogeneous Uma q-distribution and presents its distributional charachteristics.

The homogeneous Uma q-distribution can be constructed via the convex combination of $p_q^i(\alpha)$ and $g_q^i(x;\alpha,\lambda)$ for $i=I,II$ as summarized in Table 1.

Table 1. Convex combinations for the homogeneous Uma q-PDF

n	The first type of HoUma q-PDF $f_q^{I_n}(x;\lambda)$	The second type of HoUma q-PDF $f_q^{II_n}(x;\lambda)$
1	$p_q^{I(1)}g_q^{I(1)}+p_q^{I(2)}g_q^{I(2)}+p_q^{I(4)}g_q^{I(4)}$	$p_q^{II(1)}g_q^{I(1)}+p_q^{II(2)}g_q^{I(2)}+p_q^{II(4)}g_q^{I(4)}$
2	$p_q^{I(1)}g_q^{II(1)}+p_q^{I(2)}g_q^{II(2)}+p_q^{I(4)}g_q^{II(4)}$	$p_q^{II(1)}g_q^{II(1)}+p_q^{II(2)}g_q^{II(2)}+p_q^{II(4)}g_q^{II(4)}$

All variants of the homogeneous Uma q-distribution, $f_q^{i_n}(x;\lambda)(i=,II,n=1,2)$, are presented in Table 2, with normalizing constants a_q^I and a_q^{II}.

Table 2. The two types of homogeneous Uma q-PDF

n	The first type of HoUma q-PDF $f_q^{I_n}(x;\lambda)$	The second type of HoUma q-PDF $f_q^{II_n}(x;\lambda)$
1	$a_q^I\left(1+x+x^3\right)E_q^{-q\lambda x}$	$a_q^{II}\left(1+q^{-1}x+q^{-6}x^3\right)E_q^{-q\lambda x}$
2	$a_q^I\left(1+qx+q^6x^3\right)e_q^{-\lambda x}$	$a_q^{II}\left(1+x+x^3\right)e_q^{-\lambda x}$

The graphs below illustrate the homogeneous Uma q-PDF for $\lambda=0.5$ and $\lambda=2$ with varying values of the q-parameter as outlined in Table 2.

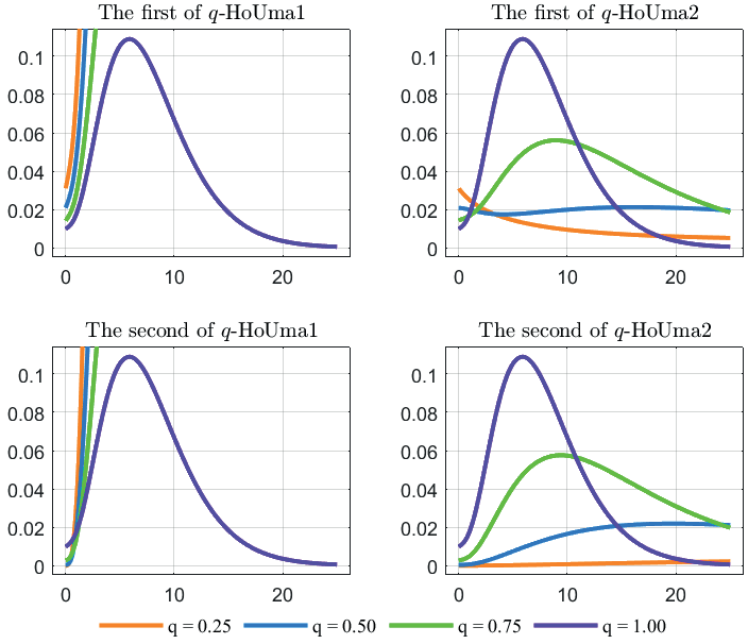

Figure 1. Graphs of the homogeneous Uma q-PDF for $\lambda = 0.5$ *and varying values of the q-parameter*

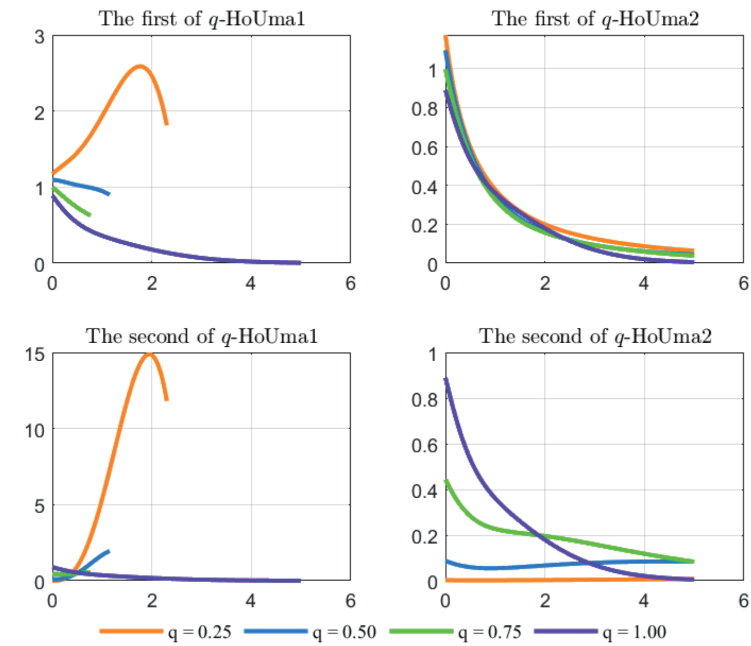

Figure 2. Graphs of the homogeneous Uma q-PDF for $\lambda = 2$ *and varying values of the q-parameter*

The plots above illustrate two types of q-HoUma distributions (for $m=1,2$), representing the q-PDFs of the homogeneous q-Uma distribution. It is observed that increasing the parameter q enhances the convexity of the model in the case of q-HoUma1, while it increases the concavity in the case of q-HoUma2. Moreover, all q-Uma distributions converge to their standard form as $q \to 1$.

The first type of homogeneous Uma q-CDF, q-moment, and q-related measures are presented as outlined in Table 3.

Table 3. The first type of homogeneous Uma q-CDF, q-Moment and related measures

n	The first type of HoUma q-CDF $F_q^{I_n}(x;\lambda)$
1	$1-\left\{1+\frac{a_q^I}{\lambda^4}\left[(\lambda\tau)^3+[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+\lambda^3\tau\right]\right\}E_q^{-\lambda\tau}$
2	$1-\left\{1+\frac{a_q^I}{\lambda^4}\left[q^3(\lambda\tau)^3+q[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+\lambda^3\tau\right]\right\}e_q^{-\lambda x}$
n	The first type of HoUma q-Moment $\mu_q^{(r)(I_n)}$
1	$a_q^I\left([r]_q!\lambda^{-(r+1)}+[r+1]_q!\lambda^{-(r+2)}+[r+4]_q!\lambda^{-(r+5)}\right)$
2	$a_q^I\left(q^{-\binom{r+1}{2}}[r]_q!\lambda^{-(r+1)}+q^{1-\binom{r+2}{2}}[r+1]_q!\lambda^{-(r+2)}+q^{6-\binom{r+5}{2}}[r+4]_q!\lambda^{-(r+5)}\right)$
n	The first type of HoUma q-expected value
1	$a_q^I\left(\lambda^{-2}+[2]_q!\lambda^{-3}+[5]_q!\lambda^{-6}\right)$
2	$a_q^I\left(q^{-1}\lambda^{-2}+q^{-2}[2]_q!\lambda^{-3}+q^{-9}[5]_q!\lambda^{-6}\right)$
n	The first type of HoUma q-variance
1	$a_q^I\left([2]_q!\lambda^{-3}+[3]_q!\lambda^{-4}+[6]_q!\lambda^{-7}\right)-\left(a_q^I\right)^2\left(\lambda^{-2}+[2]_q!\lambda^{-3}+[5]_q!\lambda^{-6}\right)^2$
2	$a_q^I\left(q^{-1}[2]_q!\lambda^{-3}+q^{-2}[3]_q!\lambda^{-4}+q^{-9}[6]_q!\lambda^{-7}\right)-\left(a_q^I\right)^2\left(q^{-1}\lambda^{-2}+q^{-2}[2]_q!\lambda^{-3}+q^{-9}[5]_q!\lambda^{-6}\right)^2$

As shown in Table 4, the second type of homogeneous Uma q-CDF, q-moment, and q-related measures are given by:

Table 4. The second type of homogeneous Uma q-CDF, q-Moment and related measures

n	The second type of HoUma q-CDF $F_q^{II_n}(x;\lambda)$
1	$1-\left\{1+\frac{a_q^{II}}{\lambda^4}\left[q^{-6}\left((\lambda\tau)^3+[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau\right)+q^{-1}\lambda^3\tau\right]\right\}E_q^{-\lambda\tau}$
2	$1-\left\{1+\frac{a_q^{II}}{\lambda^4}\left[q^{-6}\left(q^3(\lambda\tau)^3+q[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau\right)+q^{-1}\lambda^3\tau\right]\right\}e_q^{-\lambda\tau}$
n	The second type of HoUma q-Moment $\mu_q^{(r)(II_n)}$
1	$a_q^{II}\left([r]_q!\lambda^{-(r+1)}+q^{-1}[r+1]_q!\lambda^{-(r+2)}+q^{-6}[r+4]_q!\lambda^{-(r+5)}\right)$
2	$a_q^{II}\left(q^{-\binom{r+1}{2}}[r]_q!\lambda^{-(r+1)}+q^{-\binom{r+2}{2}}[r+1]_q!\lambda^{-(r+2)}+q^{-\binom{r+5}{2}}[r+4]_q!\lambda^{-(r+5)}\right)$
n	The second type of HoUma q-expected value
1	$a_q^{II}\left(\lambda^{-2}+q^{-1}[2]_q!\lambda^{-3}+q^{-6}[5]_q!\lambda^{-6}\right)$
2	$a_q^{II}\left(q^{-1}\lambda^{-2}++q^{-3}[2]_q!\lambda^{-3}+q^{-15}[5]_q!\lambda^{-6}\right)$
n	The second type of q-variance
1	$a_q^{II}\left([2]_q!\lambda^{-3}+q^{-1}[3]_q!\lambda^{-4}+q^{-6}[6]_q!\lambda^{-7}\right)-\left(a_q^{II}\right)^2\left(\lambda^{-2}+q^{-1}[2]_q!\lambda^{-3}+q^{-6}[5]_q!\lambda^{-6}\right)^2$
2	$a_q^{II}\left(q^{-1}[2]_q!\lambda^{-3}++q^{-3}[3]_q!\lambda^{-4}+q^{-15}[6]_q!\lambda^{-7}\right)-\left(a_q^{II}\right)^2\left(q^{-1}\lambda^{-2}++q^{-3}[2]_q!\lambda^{-3}+q^{-15}[5]_q!\lambda^{-6}\right)^2$

Hybrid Type of the Uma q-distribution

This section introduces the hybrid Uma q-distribution and presents its distributional charachteristics.

The hybrid Uma q-distribution can be constructed via the convex combination of $p_q^i(\alpha)$ and $g_q^i(x;\alpha,\lambda)$ for $i=I,II$ as summarized in Table 5.

Table 5. Convex combinations for the hybrid Uma q-PDFs

n	The first type of HyUma q-PDF $f_q^{I_n}(x;\lambda)$	The second type of HyUma q-PDF $f_q^{II_n}(x;\lambda)$
1	$p_q^{I(1)}g_q^{I(1)}+p_q^{I(2)}g_q^{I(2)}+p_q^{I(4)}g_q^{II(4)}$	$p_q^{II(1)}g_q^{I(1)}+p_q^{II(2)}g_q^{I(2)}+p_q^{II(4)}g_q^{II(4)}$
2	$p_q^{I(1)}g_q^{I(1)}+p_q^{I(2)}g_q^{II(2)}+p_q^{I(4)}g_q^{I(4)}$	$p_q^{II(1)}g_q^{I(1)}+p_q^{II(2)}g_q^{II(2)}+p_q^{II(4)}g_q^{I(4)}$
3	$p_q^{I(1)}g_q^{I(1)}+p_q^{I(2)}g_q^{II(2)}+p_q^{I(4)}g_q^{II(4)}$	$p_q^{II(1)}g_q^{I(1)}+p_q^{II(2)}g_q^{II(2)}+p_q^{II(4)}g_q^{II(4)}$
4	$p_q^{I(1)}g_q^{II(1)}+p_q^{I(2)}g_q^{I(2)}+p_q^{I(4)}g_q^{II(4)}$	$p_q^{II(1)}g_q^{II(1)}+p_q^{II(2)}g_q^{I(2)}+p_q^{II(4)}g_q^{II(4)}$
5	$p_q^{I(1)}g_q^{II(1)}+p_q^{I(2)}g_q^{II(2)}+p_q^{I(4)}g_q^{I(4)}$	$p_q^{II(1)}g_q^{II(1)}+p_q^{II(2)}g_q^{II(2)}+p_q^{II(4)}g_q^{I(4)}$
6	$p_q^{I(1)}g_q^{II(1)}+p_q^{I(2)}g_q^{I(2)}+p_q^{I(4)}g_q^{I(4)}$	$p_q^{II(1)}g_q^{II(1)}+p_q^{II(2)}g_q^{I(2)}+p_q^{II(4)}g_q^{I(4)}$

All variants of the hybrid Uma q-PDF, $f_q^{i_n}(x;\lambda)(i=,II,n=1,2,\ldots,6)$, are presented in Table 2, with normalizing constants a_q^I and a_q^{II}.

Table 6. The hybrid Uma q-PDFs

n	The first type of HyUma q-PDF $f_q^{I_n}(x;\lambda)$	The second type of HyUma q-PDF $f_q^{II_n}(x;\lambda)$
1	$a_q^I\left\{(1+x)E_q^{-q\lambda x}+q^6x^3e_q^{-\lambda x}\right\}$	$a_q^{II}\left\{(1+q^{-1}x)E_q^{-q\lambda x}+x^3e_q^{-\lambda x}\right\}$
2	$a_q^I\left\{(1+x^3)E_q^{-q\lambda x}+qxe_q^{-\lambda x}\right\}$	$a_q^{II}\left\{(1+q^{-6}x^3)E_q^{-q\lambda x}+xe_q^{-\lambda x}\right\}$
3	$a_q^I\left\{E_q^{-q\lambda x}+(qx+q^6x^3)e_q^{-\lambda x}\right\}$	$a_q^{II}\left\{E_q^{-q\lambda x}+(x+x^3)e_q^{-\lambda x}\right\}$
4	$a_q^I\left\{xE_q^{-q\lambda x}+(1+q^6x^3)e_q^{-\lambda x}\right\}$	$a_q^{II}\left\{q^{-1}xE_q^{-q\lambda x}+(1+x^3)e_q^{-\lambda x}\right\}$
5	$a_q^I\left\{x^3E_q^{-q\lambda x}+(1+x)e_q^{-\lambda x}\right\}$	$a_q^{II}\left\{q^{-6}x^3E_q^{-q\lambda x}+(1+x)e_q^{-\lambda x}\right\}$
6	$a_q^I\left\{(x+x^3)E_q^{-q\lambda x}+e_q^{-\lambda x}\right\}$	$a_q^{II}\left\{(q^{-1}x+q^{-6}x^3)E_q^{-q\lambda x}+e_q^{-\lambda x}\right\}$

The hybrid configurations of the Uma q-PDF for $\lambda = 0.5$ and $\lambda = 2$ with varying values of the q-parameter as outlined in Table 3 and 5. are depicted in Figures 2 and 3.

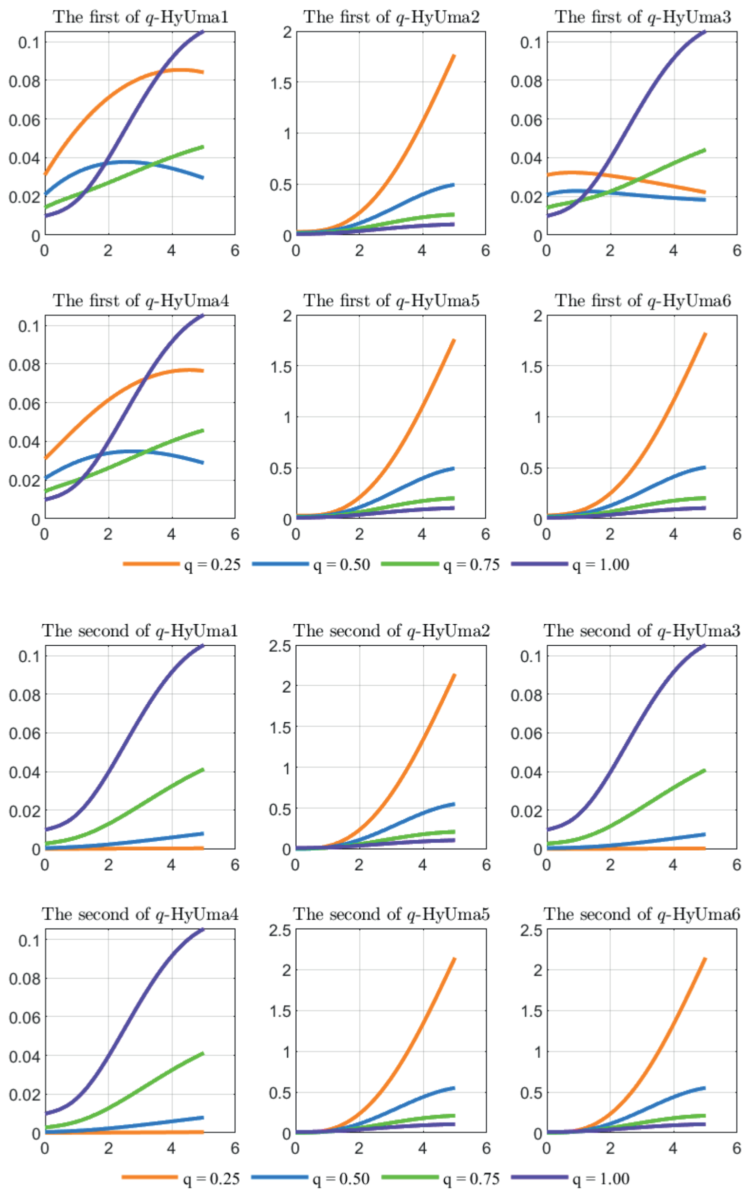

Figure 2. Graphs of the hybrid Uma q-PDF for $\lambda = 0.5$ ***and varying values of the q-parameter***

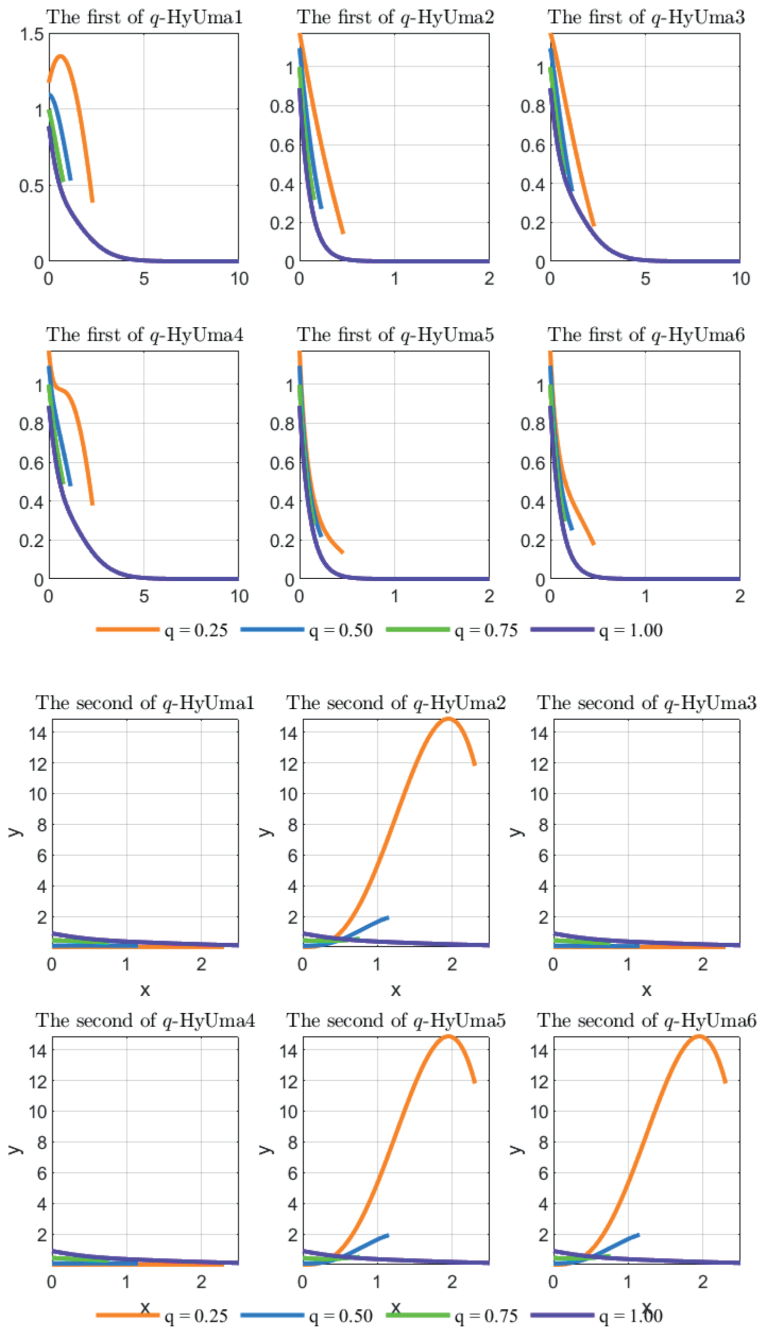

Figure 3. Graphs of the hybrid Uma q-PDF for $\lambda = 2$ ***and varying values of the q-parameter***

Two hybrid q-Uma models, each consisting of six q-PDFs, are introduced. The 2nd, 5th, and 6th q-PDFs exhibit similar patterns, while the 1st, 3rd, and 4th form another group. This similarity arises from the multiplication of the q-exponential function $E_q^{-q\lambda x}$ by x^3.

The two types of homogeneous Uma q-CDF are presented as outlined in Table 4.

Table 4. The hybrid Uma q-CDFs

n	The first type of HoUma q-CDF $F_q^{I_n}(x;\lambda)$
1	$1-\frac{a_q^I}{\lambda^4}\left\{\left[\lambda^3+\lambda^2(1+\lambda\tau)\right]E_q^{-\lambda\tau}+\left[q^3(\lambda\tau)^3+q[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!\right]e_q^{-\lambda\tau}\right\}$
2	$1-\frac{a_q^I}{\lambda^4}\left\{\lambda^2(1+q\lambda\tau)e_q^{-\lambda\tau}+\left[(\lambda\tau)^3+[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!+\lambda^3\right]E_q^{-\lambda\tau}\right\}$
3	$1-\frac{a_q^I}{\lambda^4}\left\{\lambda^3E_q^{-\lambda\tau}+\left[q^3(\lambda\tau)^3+q[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!+\lambda^2(1+\lambda\tau)\right]e_q^{-\lambda\tau}\right\}$
4	$1-\frac{a_q^I}{\lambda^4}\left\{\lambda^2(1+\lambda\tau)E_q^{-\lambda\tau}+\left[q^3(\lambda\tau)^3+q[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!+\lambda^3\right]e_q^{-\lambda\tau}\right\}$
5	$1-\frac{a_q^I}{\lambda^4}\left\{\left[\lambda^3+\lambda^2(1+q\lambda\tau)\right]e_q^{-\lambda\tau}+\left[(\lambda\tau)^3+[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!\right]E_q^{-\lambda\tau}\right\}$
6	$1-\frac{a_q^I}{\lambda^4}\left\{\lambda^3e_q^{-\lambda\tau}+\left[(\lambda\tau)^3+[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!+\lambda^2(1+\lambda\tau)\right]E_q^{-\lambda\tau}\right\}$
n	The second type of HoUma q-CDF $F_q^{II_n}(x;\lambda)$
1	$1-\frac{a_q^{II}}{\lambda^4}\left\{\left[\lambda^3+q^{-1}\lambda^2(1+\lambda\tau)\right]E_q^{-\lambda\tau}+q^{-6}\left[q^3(\lambda\tau)^3+q[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!\right]e_q^{-\lambda\tau}\right\}$
2	$1-\frac{a_q^{II}}{\lambda^4}\left\{q^{-1}\lambda^2(1+q\lambda\tau)e_q^{-\lambda\tau}+\left[q^{-6}\left((\lambda\tau)^3+[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!\right)+\lambda^3\right]E_q^{-\lambda\tau}\right\}$
3	$1-\frac{a_q^{II}}{\lambda^4}\left\{\lambda^3E_q^{-\lambda\tau}+\left[q^{-6}\left(q^3(\lambda\tau)^3+q[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!\right)+q^{-1}\lambda^2(1+\lambda\tau)\right]e_q^{-\lambda\tau}\right\}$
4	$1-\frac{a_q^{II}}{\lambda^4}\left\{q^{-1}\lambda^2(1+\lambda\tau)E_q^{-\lambda\tau}+\left[q^{-6}\left(q^3(\lambda\tau)^3+q[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!\right)+\lambda^3\right]e_q^{-\lambda\tau}\right\}$
5	$1-\frac{a_q^{II}}{\lambda^4}\left\{\left[\lambda^3+q^{-1}\lambda^2(1+q\lambda\tau)\right]e_q^{-\lambda\tau}+q^{-6}\left[(\lambda\tau)^3+[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!\right]E_q^{-\lambda\tau}\right\}$

n	The first type of HoUma q-CDF $F_q^{I_n}(x;\lambda)$
1	$1-\frac{a_q^I}{\lambda^4}\left\{\left[\lambda^3+\lambda^2(1+\lambda\tau)\right]E_q^{-\lambda\tau}+\left[q^3(\lambda\tau)^3+q[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!\right]e_q^{-\lambda\tau}\right\}$
6	$1-\frac{a_q^{II}}{\lambda^4}\left\{\lambda^3 e_q^{-\lambda\tau}+\left[q^{-6}\left((\lambda\tau)^3+[3]_q(\lambda\tau)^2+[3]_q!\lambda\tau+[3]_q!\right)+q^{-1}\lambda^2(1+\lambda\tau)\right]E_q^{-\lambda\tau}\right\}$

Furthermore, the derivation of the hybrid q-moment and the corresponding statistical measures can be carried out in an analogous manner for the homogeneous type.

DISCUSSION AND CONCLUSION

The evolution of q-distributions represents a natural progression in the development of q-calculus. q-calculus serves as a parametric generalization of classical calculus, with the classical framework being recovered in the limit as $q\to1$.

In this paper, we introduce the Uma q-distribution, and and provide a detailed analysis of the structural and statistical properties, including its modeling, shape, moment of the specific cases corresponding to the homogeneous and hybrid types.

Our findings suggest that the proposed q-distribution holds significant promise and may have widespread applications across various fields. In future research, we aim to explore the finite mixture and compound q-distribution.

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Conflict of Interest

The authors have declared that there is no conflict of interest.

Author Contributions

Both authors contributed equally to the finalization of this paper.